MTA Pre-Regional Mathematical Olympiad (PRMO),2019

Date: August 25, 2019 **Time: 10 AM to 1 PM**

Number of Questions 30: Max Marks: 102

INSTRUCTIONS

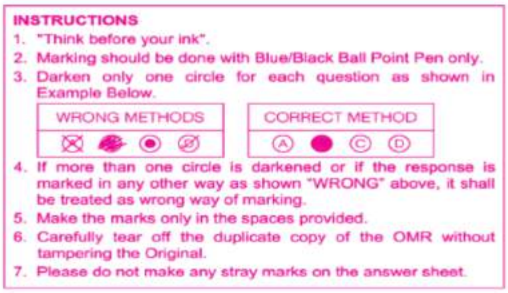
1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a black pencil or a black or blue ball pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your PRMO score.

4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be cosidered as your final answer.

7. Questions 1 to 6 carry 2 marks each; questions 7 to 21 carry 3 marks each; questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

1. Consider the sequence of numbers $[n + \sqrt{2n} + \frac{1}{2}]$ for $n \geq 1$, where $[x]$ denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < \cdots$ then find n_{12}.

2. If $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers, what is the value of $|a + b + c + d|$?

3. Find the number of positive integers less than 101 that *can not* be written as the difference of two squares of integers.

4. Let $a_1 = 24$ and form the sequence a_n, $n \geq 2$ by $a_n = 100a_{n-1} + 134$. The first few terms are

$$24, 2534, 253534, 25353534, \ldots$$

What is the least value of n for which a_n is divisible by 99?

5. Let N be the smallest positive integer such that $N + 2N + 3N + \cdots + 9N$ is a number all whose digits are equal. What is the sum of the digits of N?

6. Let ABC be a triangle such that $AB = AC$. Suppose the tangent to the circumcircle of ABC at B is perpendicular to AC. Find $\angle ABC$ measured in degrees.

SPACE FOR ROUGH WORK

7. Let $s(n)$ denote the sum of the digits of a positive integer n in base 10. If $s(m) = 20$ and $s(33m) = 120$, what is the value of $s(3m)$?

8. Let $F_k(a, b) = (a + b)^k - a^k - b^k$ and let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. For how many ordered pairs (a, b) with $a, b \in S$ and $a \leq b$ is $\frac{F_5(a, b)}{F_3(a, b)}$ an integer?

9. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is α° and the smaller one β° then what is the value of $\alpha - \beta$?

10. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two digit number. When I returned, it was y minutes past 6'O clock, and I noticed that (i) I walked exactly for x minutes and (ii) y was a 2 digit number obtained by reversing the digits of x. How many minutes did I walk?

11. Find the largest value of a^b such that the positive integers $a, b > 1$ satisfy $a^b b^a + a^b + b^a = 5329$.

SPACE FOR ROUGH WORK

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set

$$\{10a + b : 1 \leq a \leq 5, 1 \leq b \leq 5\}$$

where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

13. Consider the sequence

$$1, 7, 8, 49, 50, 56, 57, 343, \ldots$$

which consists of sums of distinct powers of 7, that is, $7^0, 7^1, 7^0 + 7^1, 7^2, \ldots$, in increasing order. At what position will 16856 occur in this sequence?

14. Let $\mathcal{R}$ denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ divide $\mathcal{R}$ into four regions $\mathcal{R}_i$, $i = 1, 2, 3, 4$. If $|\mathcal{R}_i|$ denotes the area of the region $\mathcal{R}_i$ and if $[\mathcal{R}_1] > [\mathcal{R}_2] > [\mathcal{R}_3] > [\mathcal{R}_4]$, determine $[\mathcal{R}_1] - [\mathcal{R}_2] - [\mathcal{R}_3] + [\mathcal{R}_4]$. (Here $[\Omega]$ denotes the area of the region Ω in the plane.)

15. In base -2 notation, digits are 0 and 1 only and the places go up in powers of -2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base -2 how many non zero digits does it contain?

16. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N?

SPACE FOR ROUGH WORK

17. Let a, b, c be distinct positive integers such that $b+c-a$, $c+a-b$ and $a+b-c$ are all perfect squares. What is the largest possible value of $a+b+c$ smaller than 100?

18. What is the smallest prime number p such that p^3+4p^2+4p has exactly 30 positive divisors?

19. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer?

20. How many 4-digit numbers $\overline{abcd}$ are there such that $a < b < c < d$ and $b-a < c-b < d-c$?

21. Consider the set E of all positive integers n such that when divided by 9,10,11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.

SPACE FOR ROUGH WORK

22. In parallelogram $ABCD$, $AC = 10$ and $BD = 28$. The points K and L in the plane of $ABCD$ move in such a way that $AK = BD$ and $BL = AC$. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

23. Let t be the area of a regular pentagon with each side equal to 1. Let $P(x) = 0$ be the polynomial equation with least degree, having integer coefficients, satisfied by $x = t$ and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of $P(x)$, What is the integer closest to $\sqrt{M}$? $(\sin 18^\circ = (\sqrt{5} - 1)/2.)$

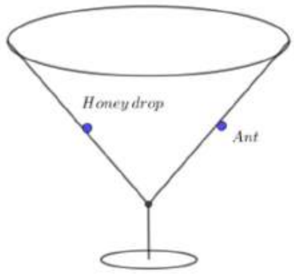
24. For $n \geq 1$, let a_n be the number beginning with n 9's followed by 744; e.g., $a_4 = 9999744$. Define $f(n) = \max\{m \in \mathbb{N} \,|\, 2^m \text{ divides } a_n\}$, for $n \geq 1$. Find $f(1) + f(2) + f(3) + \cdots + f(10)$.

25. Let ABC be an isosceles triangle with $AB = BC$. A trisector of $\angle B$ meets AC at D. If AB, AC and BD are integers and $AB - BD = 3$, find AC.

26. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance traveled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N?

SPACE FOR ROUGH WORK

27. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass. (See the figure.) If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d? (Ignore the thickness of the glass.)



28. In a triangle ABC, it is known that $\angle A = 100^\circ$ and $AB = AC$. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that $\sin 10^\circ \approx 0.174$.

29. Let ABC be an acute angled triangle with $AB = 15$ and $BC = 8$. Let D be a point on AB such that $BD = BC$. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible values of AE, find $[\alpha]$ the integer part of α.

30. For any real number x, let $[x]$ denote the integer part of x; $\{x\}$ be the fractional part of x ($\{x\} = x - [x]$). Let A denote the set of all real numbers x satisfying

$$\{x\} = \frac{x + [x] + [x + (1/2)]}{20}.$$

If S is the sum of all numbers in A, find $[S]$.

SPACE FOR ROUGH WORK