FUNDAMENTAL THEOREM OF ARITHMETIC

FACTORS AND MULTIPLES

Factors

'a' is a factor of 'b' if there exists a relation such that $a \times n = b$, where 'n' is any natural number.

- 1 is a factor of all numbers as $1 \times b = b$.
- Factor of a number cannot be greater than the number (in fact the largest factor will be the number itself).

Thus factors of any number will lie between 1 and the number itself (both inclusive) and they are limited.

Multiples

'a' is a multiple of 'b' if there exists a relation of the type $b \times n = a$. Thus the multiples of 6 are $6 \times 1 = 6$, $6 \times 2 = 12$, $6 \times 3 = 18$, $6 \times 4 = 24$, and so on.

- The smallest multiple will be the number itself and the number of multiples would be infinite.

Ex.1 How many numbers from 200 to 600 are divisible by 4, 5, 6 ?

Sol. Every such number must be divisible by L.C.M. of (4, 5, 6) = 60.

$$\left[\frac{600}{60}\right] - \left[\frac{200}{60}\right] = 10 - 3 = 7.$$

Such numbers are 240, 300, 360, 420, 480, 540 and 600.

Clearly, there are 7 such numbers.

Factorization : It is the process of splitting any number into a form where it is expressed only in terms of the most basic prime factors.

For example, $36 = 2^2 \times 3^2$. It is expressed in the factorized form in terms of its basic prime factors.

Number of factors: For any composite number C, which can be expressed as

$$C = a^p \times b^q \times c^r \times .....,$$

where a, b, c ..... are all prime factors and p, q, r are positive integers, the number of factors is equal to

$(p + 1) \times (q + 1) \times (r + 1)....$ e.g. $36 = 2^2 \times 3^2$.

So the factors of 36 = $(2 + 1) \times (2 + 1) = 3 \times 3 = 9$.

Ex.2 If $N = 12^3 \times 3^4 \times 5^2$, find the total number of even factors of N.

Sol. The factorized form of N is

$(2^2 \times 3^1)3 \times 3^4 \times 5^2 \Rightarrow 2^6 \times 3^7 \times 5^2$.

Hence, the total number of factors of N is $(6 + 1)(7 + 1)(2 + 1) = 7 \times 8 \times 3 = 168$.

Some of these are odd multiples and some are even. The odd multiples are formed only with the combination of 3s and 5s.

So, the total number of odd factors is $(7 + 1)(2 + 1) = 24$.

Therefore, the number of even factors is $168 - 24 = 144$.

Ex.3 A number N when factorized can be written $N = a^4 \times b^3 \times c^7$. Find the number of perfect squares which are factors of N (The three prime numbers a, b, c > 2).

Sol. In order that the perfect square divides N, the powers of 'a' can be 0, 2 or 4, i.e. 3. Powers of 'b' can be 0, 2, i.e. 2. Power of 'c' can be 0, 2, 4 or 6, i.e. 4.

Hence, a combination of these powers given 3 × 2 × 4 i.e. 24 numbers.

So, there are 24 perfect squares that divides N.

Fundamental Theorem of Arithmetic

Every composite number can be expressed as a product of primes, and this factorization is unique, except for the order in which the prime factors occurs.

HCF and LCM of numbers

HCF and LCM of numbers can be determined by prime factorization. This is nothing but an application of the fundamental theorem of arithmetic.

HCF = Product of the smallest power of each common factor.

LCM = Product of the biggest power of each prime factor

Let a and b be natural numbers.

Then their HCF × LCM = a × b

Remark: LCM is always divisible by HCF.

(i) A number on being divided by d_1 and d_2 successively leaves the remainders r_1 and r_2, respectively. If the number is divided by $d_1 \times d_2$, then the remainder is $(d_1 \times r_3 + r_1)$.

Ex.4 A number on being divided by 10 and 11 successively leaves the remainder 5 and 7, respectively. Find the remainder when the same number of divided by 110.

Sol. The required remainder

$= d_1 \times r_2 + r_1 = 10 \times 7 + 5 = 75$.

(ii) To find the number of numbers divisible by a certain integer.

Ex.5 How many numbers up to 532 are divisible by 15?

Sol. We divide 532 by 15.

532 = <u>35</u> × 15 + 7

The quotient obtained is the required number of numbers. Thus there are 35 such numbers.

Ex.6 How many numbers up to 300 are divisible by 5 and 7 together?

Sol. LCM of 5 and 7 = 35

We divide 300 by 35

300 = <u>8</u> × 35 + 20

Thus there are 8 such numbers.

(iii) Two numbers when divided by a certain divisor give remainders r_1 and r_2. When their sum is divided by the same divisor, the remainder is r_3.

The divisor is given by $r_1 + r_2 – r_3$.

Ex.7 Two numbers when divided by a certain divisor give remainders 437 and 298, respectively.

When their sum is divided by the same divisor, the remainder is 236. Find the divisor.

Sol. The required divisor

= 437 + 298 – 236 = 499.

(iv) Product of two numbers = LCM of the numbers × HCF of the numbers.

Ex.8 The HCF and the LCM of any two numbers are 63 and 1260, respectively. If one of the two numbers is 315, find the other number.

Sol. The required number

$$= \frac{\text{LCM} \times \text{HCF}}{\text{First number}} = \frac{1260 \times 63}{315} = 252$$

(v) To find the greatest number that will exactly divide x, y and z.

Required number = HCF of x, y, and z. e.g. Find the greatest number that will exactly divide 200 and 320.

Sol. The required greatest number = HCF of 200 and 320 = 40.

(vi) To find the greatest number that will divide x, y, and z leaving remainders a, b, and c, respectively.

Required number = HCF of (x – a), (y – b) and (z – c).

Ex.9 Find the greatest number that will divide 148, 246 and 623 leaving remainders 4, 6 and 11, respectively.

Sol. The required greatest number = HCF of (148 – 4), (246 – 6) and (623 – 11),

i.e. HCF of 144, 240 and 612 = 12.

(vii) To find the least number which is exactly divisible by x, y and z.

Required number = LCM of x, y and z.

e.g. What is the smallest number which is exactly divisible by 36, 45, 63 and 80?

Sol. The required smallest number = LCM of 36, 45, 63 and 80 = 5040.

(viii) To find the least number which when divided by x, y and z leaves the remainders a, b and c respectively, such that (x – a) = (y – b) = (z – c) = k (say).

∴ Required number = (LCM of x, y and z) – k.

Ex.10 Find the least number which when divided by 36, 48 and 64 leaves the remainders 25, 37 and 53, respectively.

Sol. Since (36 – 25) – (48 – 37) = (64 – 53) = 11, therefore the required smallest number (LCM of 36, 48 and 64) –11 = 576 – 11 = 565.

(ix) To find the least number which when divided by x, y and z leaves the same remainder r in each case.

Required number = (LCM of x, y, and z)+r.

Ex.11 Find the least number which when divided by 12, 16 and 18, will leave a remainder 5 in each case.

Sol. The required smallest number = (LCM of 12, 16 and 18) + 5 = 144 + 5 = 149.

(x) To find the greatest number that will divide x, y and z leaving the same remainder in each case.

(a) When the value of remainder r is given:

Required number = HCF of (x – r), (y – r) and (z – r).

(b) When the value of reminder is not given:
Required number = HCF of |x – y|, |y – z| and |z – x|.

Ex.12 Find the greatest number which will divide 772 and 2778 so as to leave the remainder 5 in each case.

Sol. The required greatest number = HCF of (772 – 5) and (2778 – 5) = HCF of 767 and 2773 = 59.

Ex.13 Find the greatest number which on dividing 152, 277 and 427 leaves same remainder.

Sol. The required greatest number.
= HCF of |x – y|, |y – z| and |z – x|
= HCF of |152 – 277|, |277 – 427| and |427 – 152|
= HCF of 125, 150 and 275 = 25.

(xi) To find the n-digit greatest number which, when divided by x, y and z

(a) leaves no remainder (i.e., exactly divisible)

Step-1 : LCM of x, y and z = L

Step-2 : $\frac{L\overline{)\text{n digit greatest number}}}{\text{Remainder} = R}$

Step-3: Required number = n-digit greatest number – R

(b) leaves reminder K in each case
Required number = (n-digit greatest number – R) + K.

Ex.14 Find the greatest number of 4 digits which, when divided by 12, 18, 21 and 28, leaves 3 as a remainder in each case.

Sol. LCM of 12, 18. 21 and 28 = 252.

$$252\overline{)9999}(39$$
$$9828$$
$$\overline{171}$$

∴ The required number = (9999 – 171) + 3 = 9931.

(xii) To find the n-digit smallest number which when divided by x, y and z

(a) leaves no remainder (i.e. exactly divisible)

Step-1 : LCM of x, y and z = L

Step-2 : $\frac{L\overline{)\text{n digit greatest number}}}{\text{Remainder} = R}$

Step-3 : Required number = n-digit smallest number + (L – R).

(b) leaves remainder K in each case. Required number = n-digit smallest number + (L – R) + K.

Ex.15

(a) Find the least number of four digits which is divisible by 4, 6. 8 and 10.

Sol. LCM of 4, 6, 8, and 10 = 120.

$$120\overline{)1000}(8$$
$$960$$
$$\overline{40}$$

∴ The required number = 1000 + (120 – 40) = 1080.

(b) Find the smallest 4-digit number, such that when divided by 12, 18, 21 and 28, it leaves remainder 3 in each case.

Sol. LCM of 12. 18, 21 and 28 = 252.

$$252\overline{)1000}(3$$
$$\underline{756}$$
$$\underline{244}$$

$\therefore$ The required number = 1000 + (252 – 244) + 3 = 1011.

(xiii) HCF and LCM of fractions

HCF of

$$= \frac{a}{b}, \frac{c}{d}, \frac{e}{f} = \frac{\text{HCF of numerators}}{\text{LCM of denominators}}$$

$$= \frac{\text{HCF}(a, c, e)}{\text{LCM}(b, d, f)}$$

LCM of

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{f} = \frac{\text{LCM of numerators}}{\text{HCF of denominators}}$$

$$= \frac{\text{LCM}(a, c, e)}{\text{HCF}(b, d, f)}$$

(xiv) For any three positive integers p, q, r HCF (p, q, r) × LCM (p, q, r) $\neq$ p × q × r. However, the following result hold good:

Ex.16 Given that HCF (306, 657) = 9. Find LCM (306, 657).

Sol: HCF (306, 657) = 9 means HCF of 306 and 657 = 9

Required LCM (306, 657) means required LCM of 306 and 657.

For any two positive integers:

their LCM (a, b)

$$= \frac{\text{Product of the numbers}}{\text{HCF}(a, b)}$$

i.e., LCM (306, 657) = $\frac{306 \times 657}{9}$

= 22,338

Ex.17 In a morning walk three persons step off together, their steps measure 80 cm, 85 cm and 90 cm respectively.

What is the minimum distance each should walk so that they can cover the distance in complete steps?

Sol: Required minimum distance each should walk so, that they can cover the distance in complete step is the L.C.M. of 80 cm, 85 cm and 90 cm

80 = 24 × 5

85 = 5 × 17

90 = 2 × 32 × 5

LCM = 24 × 32 × 51 × 171

LCM = 16 × 9 × 5 × 17

LCM = 12240 cm = 122 m 40 cm.

Ex.18 Find the greatest number of six digits exactly divisible by 15, 24 and 36.

Sol: We have :

15 = 3 × 5; 24 = 23 × 3; 36 = 22 × 32

LCM (15, 24, 36) = 23 × 32 × 5 = 8 × 9 × 5 = 360

Now the greatest six-digit number is 999999.

If we divide 999999 by 360, then q = 2777 and r = 279

$\therefore$ Required number = 999999 – 279

= 999720.

WORK SHEET

1. Find the largest four digit number which when reduced by 54, is perfectly divisible by all even natural numbers less than 20.

2. $12^3 \times 3^4 \times 5^2$, find the total number of even factors of N.

3. When simplified, the product

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)....\left(1-\frac{1}{n}\right) \text{ equals}$$

(A) $\frac{1}{n}$ (B) $\frac{2}{n}$

(C) $\frac{2(n-1)}{n}$ (D) $\frac{2}{n(n+1)}$

4. Find the greatest number that will divide 43, 91 and 183 so as to leave the same remainder in each case.

(A) 4 (B) 7

(C) 9 (D) 13

5. The H.C.F. of two numbers is 23 and the other two factors of their L.C.M. are 13 and 14. The larger of the two numbers is:

(A) 276 (B) 299

(C) 322 (D) 345

6. Six bells commence tolling together and toll at intervals of 2, 4, 6, 8 10 and 12 seconds respectively. In 30 minutes, how many times do they toll together ?

(A) 4 (B) 10

(C) 15 (D) 16

7. Let N be the greatest number that will divide 1305, 4665 and 6905, leaving the same remainder in each case. Then sum of the digits in N is:

(A) 4 (B) 5

(C) 6 (D) 8

8. The greatest number of four digits which is divisible by 15, 25, 40 and 75 is:

(A) 9000 (B) 9400

(C) 9600 (D) 9800

9. The product of two numbers is 4107. If the H.C.F. of these numbers is 37, then the greater number is:

(A) 101 (B) 107

(C) 111 (D) 185

10. Three number are in the ratio of 3 : 4 : 5 and their L.C.M. is 2400. Their H.C.F. is:

(A) 40 (B) 80

(C) 120 (D) 200

11. The G.C.D. of 1.08, 0.36 and 0.9 is:

(A) 0.03 (B) 0.9

(C) 0.18 (D) 0.108

12. The product of two numbers is 2028 and their H.C.F. is 13. The number of such pairs is:

(A) 1 (B) 2

(C) 3 (D) 4

13. The least multiple of 7, which leaves a remainder of 4, when divided by 6, 9, 15 and 18 is:

(A) 74 (B) 94

(C) 184 (D) 364

14. A, B and C start at the same time in the same direction to run around a circular stadium. A completes a round in 252 seconds, B in 308 seconds and c in 198 seconds, all starting at the same point. After what time will they again at the starting point ?

(A) 26 minutes and 18 seconds

(B) 42 minutes and 36 seconds

(C) 45 minutes

(D) 46 minutes and 12 seconds

15. Three numbers which are co-prime to each other are such that the product of the first two is 551 and that of the last two is 1073. The sum of the three numbers is:

(A) 75 (B) 81

(C) 85 (D) 89

16. Which of the following has the most number of divisors?

(A) 99 (B) 101

(C) 176 (D) 182

17. Find the HCF of 252525 and 363636.

18. If $13824 = 2^a \times 3^b$ then find a and b.

19. If $p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4}$ = 113400 where $p_1 \times p_2 \times p_3 \times p_4$ are primes in ascending order and are $x_1 \times x_2 \times x_3 \times x_4$ integers, find the value of $p_1 \times p_2 \times p_3 \times p_4$ and $x_1 \times x_2 \times x_3 \times x_4$.

20. Find the LCM and HCF of 408 and 170 by applying the fundamental theorem of arithmetic.

21. Find the greatest number consisting of 6 digits which is exactly divisible by 24, 15, 36 ?

22. What is the smallest number that when divided by three numbers such as 35, 56 and 91 leaves remainder 7 in each case ?

23. Find the least number that is divisible by the first ten natural numbers.

ANSWER SHEET

Sol.1 Even natural numbers less than 20 are 2, 4, 6, 8, 1 0, 12, 14, 16, 18.

Their LCM = 2 × LCM of first 9 natural numbers = 2 × 2520 = 5040.

This happens to be the largest four-digit number divisible by all even natural numbers less than 20.

54 was subtracted from our required number to get this number.

Hence, (required number – 54) = 5040 ⇒ Required number= 5094.

Sol.2 The factorized form of N is $(2^2 \times 3^1)^3 \times 3^4 \times 5^2 \Rightarrow 2^6 \times 3^7 \times 5^2$.

Hence, the total number of factors of N is (6 + 1) (7 + 1) (2 + 1) = 7 x 8 x 3 = 168.

Some of these are odd multiples and some are even. The odd multiples are formed only with the combination of 3s and 5s.

So, the total number of odd factors is (7 + 1) (2 + 1) = 24.

Therefore, the number of even factors is 168 – 24 = 144.

Sol.3 $\left(\frac{3-1}{3}\right)\left(\frac{4-1}{4}\right)\left(\frac{5-1}{5}\right)....\left(\frac{n-1}{n}\right)$

$\Rightarrow \frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}.....\frac{n-1}{n} \Rightarrow \frac{2}{n}$

Sol.4 Option A

Explanation: Required number = H.C.F. of (91–43), (183–91) and (183 – 43) = H.C.F. of 48, 92 and 140 = 4.

Sol.5 Option C

Explanation: Clearly, the numbers are (23 x 13) and (23 x 14).

Larger number = (23 x 14) = 322.

Sol.6 Option D

Explanation: L.C.M. of 2, 4, 6, 8, 10, 12 is 120. So, the bells will toll together after every 120 seconds (2 minutes).

In 30 minutes, they will toll together 30 + 1 = 16 times.

Sol.7 Option A

Explanation:0020N = H.C.F. of (4665 – 1305), (6905 – 4665) and (6905 – 1305) = H.C.F. of 3360, 2240 and 5600 = 1120.

Sum of digits in N = (1 + 1 + 2 + 0) = 4

Sol.8 Option C

Explanation: Greatest number of 4-digits is 9999.

L.C.M. of 15, 25, 40 and 75 is 600.

On dividing 9999 by 600, the remainder is 399.

Required number (9999 – 399) = 9600

Sol.9 Option C

Explanation: Let the numbers be 37a and 37b.

Then, 37a x 37b = 4107 ⇒ ab = 3.

Now, co-primes with product 3 are (1, 3).

So, the required numbers are (37 x 1, 37 x 3) i.e., (37, 111).

Greater number = 111.

Sol.10 Option A

Explanation: Let the numbers be 3x, 4x and 5x.

Then, their L.C.M. = 60x.

So, 60x = 2400 or x = 40.

The numbers are (3 × 40), (4 × 40) and (5 × 40). Hence, required H.C.F. = 40.

Sol.11 Option C

Explanation: Given numbers are 1.08, 0.36 and 0.90. H.C.F. of 108, 36 and 90 is 18, ∴ H.C.F. of given numbers = 0.18.

Sol.12 Option B

Explanation: Let the numbers 13a and 13b.

Then, 13a × 13b = 2028 ⇒ ab = 12.

Now, the co-primes with product 12 are (1, 12) and (3, 4).

[**Note:** Two integers a and b are said to be co-prime or relatively prime if they have no common positive factor other than 1 or, equivalently, if their greatest common divisor is 1]

So, the required numbers are (13 × 1, 13 × 12) and (13 × 3, 13 × 4).

Clearly, there are 2 such pairs.

Sol.13 Option D

Explanation: L.C.M. of 6, 9, 15 and 18 is 90.

Let required number be 90k + 4, which is multiple of 7.

Least value of k for which (90k + 4) is divisible by 7 is k = 4.

∴ Required number = (90 × 4) + 4 = 364

Sol.14 Option D

Explanation: L.C.M. of 252, 308 and 198 = 2772.

So, A, B and C will again meet at the starting point in 2772 sec. i.e., 46 min. 12 sec

Sol.15 Option C

Explanation: Since the numbers are co-prime, they contain only 1 as the common factor.

Also, the given two products have the middle number in common.

So, middle number = H.C.F. of 551 and 1073 = 29;

First number = $\left(\frac{551}{29}\right)$ = 19;

Third number = $\left(\frac{1073}{29}\right)$ = 37.

∴ Required sum = (19 + 29 + 37) = 85

Sol.16 Option C

Explanation:

99 = 1 × 3 × 3 × 11

101 = 1 × 101

176 = 1 × 2 × 2 × 2 × 2 × 11

182 = 1 × 2 × 7 × 13

So, divisors of 99 are 1, 3, 9, 11, 33, .99

Divisors of 101 are 1 and 101

Divisors of 176 are 1, 2, 4, 8, 11, 16, 22, 44, 88 and 176

Divisors of 182 are 1, 2, 7, 13, 14, 26, 91 and 182.

Hence, 176 has the most number of divisors

Sol.17

5	252525
5	50505
3	10101
7	3367
13	481
	37

2	363636
2	181818
3	90909
3	30303
3	10101
7	3367
13	481
	37

$252525 = 5^2 \cdot 3 \cdot 7 \cdot 13 \cdot 37$

$363636 = 2^2 \cdot 3^3 \cdot 7 \cdot 13 \cdot 37$

H.C.F $= 3 \cdot 7 \cdot 13 \cdot 37 = 10101$

Sol.18 To find the values of a and b, we have decompose 13824 as the product of prime factors.

2	13824
2	6912
2	3456
2	1728
2	864
2	432
2	216
3	108
3	36
3	12
2	4
	2

$13824 = 2^9 \times 3^3$

Hence the value of a is 9 and b is 3.

Ans.19

2	113400
2	56700
2	28350
5	14175
5	2835
3	567
3	189
3	63
3	21
	7

$113400 = 2^3 \times 3^4 \times 5^2 \times 71$

The values of p_1, p_2, p_3 and p_4 are 2, 3, 5 and 7 respectively.

The values of x_1, x_2, x_3 and x_4 are 3, 4, 2 and 1 respectively.

Sol.20

2	408
2	204
2	102
3	51
	17

2	170
5	85
	17

$408 = 2^3 \times 3 \times 17$

$170 = 2 \times 5 \times 17$

Common factors are 2 and 17

H.C.F = 34

L.C.M $= 2^3 \times 3 \times 5 \times 17$

L.C.M = 2040

Sol.21 We will find out the LCM of 24, 15 and 36 is 360.

The greatest 6-digit number is 999999.

Now, We will divide this number by LCM of 24, 15 and 36, we will get,

999999/360 will get remainder 279.

Now 999999 – 279 = 999720.

Now we will check it out for the numbers we will get,

999720/24 = 41655

999720/15 = 66648

999720/36 = 27770

999720 is the greatest number 6-digit number divisible by 24, 15 and 36.

Sol.22 The smallest number which when divided by 35, 56 and 91 = LCM of 35, 56 and 91

$35 = 5 \times 7$

$56 = 2 \times 2 \times 2 \times 7$

$91 = 7 \times 13$

LCM = $7 \times 5 \times 2 \times 2 \times 2 \times 13 = 3640$

The smallest number that when divided by 35, 56, 91 leaves a remainder 7 in each case = 3640 + 7 = 3647.

Sol.23 first 10 natural numbers:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

L.C.M of these natural numbers:

Factors = $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 2520$

The smallest number is 2520