JEE MAIN 2024
Paper with Solution

Maths | 31th January 2024 _ Shift-1

SECTION – A

1. $\lim_{x\to 0}\frac{e^{2|\sin x|}-2|\sin x|-1}{x^2}$

(1) is equal to 1
(2) does not exist
(3) is equal to –1
(4) is equal to 2

Sol. **(4)**

$\lim_{x\to 0}\frac{e^{2|\sin x|}-2|\sin x|-1}{x^2}$

$e^{2|\sin x|} = 1 + 2\,|\sin x| + \frac{4|\sin x|^2}{2}\ldots\ldots$

$= \lim_{x\to 0}\frac{1+2|\sin x|+2|\sin x|^2\ldots.-2|\sin x|-1}{x^2}$

$=\lim_{x\to 0}\frac{2|\sin x|^2}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2$

2. Let $\vec{a}=3\hat{i}+\hat{j}-2\hat{k},\vec{b}=4\hat{i}+\hat{j}+7\hat{k}$ and $\vec{c}=\hat{i}-3\hat{j}+4\hat{k}$ be three vectors. If a vectors $\vec{p}$ satisfies $\vec{p}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{p}\cdot\vec{a}=0$, then $\vec{p}\cdot(\hat{i}-\hat{j}-\hat{k})$ is equal to

(1) 24
(2) 36
(3) 32
(4) 28

Sol. **(3)**

$\vec{p}\times\vec{b} = \vec{c}\times\vec{b}$

$(\vec{p}-\vec{c})\times\vec{b} = 0$

$\vec{p}-\vec{c} = \lambda\,\vec{b}$

$\vec{p} = \vec{c} + \lambda\vec{b}$

$\vec{p}\,.\,\vec{a} = \vec{c}\,.\,\vec{a} + \lambda\,(\vec{b}\,.\,\vec{a}) \Rightarrow \vec{c}\cdot\vec{a}=-\lambda(\vec{b}\cdot\vec{a})\ (\because \vec{p}.\vec{a})=0$

$\lambda = -\frac{\vec{c}.\vec{a}}{\vec{b}.\vec{a}}$

$\lambda = -8$

$\vec{p} = \vec{c} - 8\,\vec{b}$

$\vec{p} = (\hat{i} - 3\hat{j} + 4k) - 8\,(4\hat{i} + \hat{j} + 7k)$

$\vec{p} = -31\hat{i} - 11\hat{j} - 52k$

$\vec{p} = -(31\hat{i} + 11\hat{j} + 52k)$

Now $\vec{p}.(\hat{i}-\hat{j}-k) = -(31\hat{i} + 11\hat{j} + 52k)(\hat{i}-\hat{j}-k)$

$= -(31 - 11 - 52)$

$= 32$

3. If $f(x)=\begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 3x^2+2 & 2x & x^3+6 \\ x^3-x & 4 & x^2-2 \end{vmatrix}$ for all $x\in\mathbb{R}$, then $2f(0)+f'(0)$ is equal to

(1) 18 (2) 24 (3) 48 (4) 42

Sol. **(4)**

$$f(x) = \begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 3x^2+2 & 2x & x^3+6 \\ x^3-x & 4 & x^2-2 \end{vmatrix}$$

$$f'(x) = \begin{vmatrix} 3x^2 & 4x & 3 \\ 3x^2+2 & 2x & x^3+6 \\ x^3-x & 4 & x^2-2 \end{vmatrix} + \begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 6x & 2 & 3x^2 \\ x^3-x & 4 & x^2-2 \end{vmatrix} + \begin{vmatrix} x^3 & 2x^2+1 & 1+3x \\ 3x^2+2 & 2x & x^3+6 \\ 3x^2-1 & 0 & 2x \end{vmatrix}$$

$$2f(0)+f'(0) = 2\begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & 4 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 2 & 0 & 6 \\ 0 & 4 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 6 \\ -1 & 0 & 0 \end{vmatrix}$$

$= 24 + 24 + 0 - 6$

$= 42$

4. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx}=\frac{(\tan x)+y}{\sin x(\sec x-\sin x\tan x)}, x\in\left(0,\frac{\pi}{2}\right)$ satisfying the condition $y\left(\frac{\pi}{4}\right)=2$. Then, $y\left(\frac{\pi}{3}\right)$ is

(1) $\frac{\sqrt{3}}{2}(2+\log_e 3)$ (2) $\sqrt{3}\left(2+\log_e\sqrt{3}\right)$ (3) $\sqrt{3}(2+\log_e 3)$ (4) $\sqrt{3}(1+2\log_e 3)$

Sol. **(2)**

$$\frac{dy}{dx}=\frac{\tan x + y}{\sin x(\sec x-\sin x\tan x)}$$

$$\frac{dy}{dx}=\frac{\sin x + y\cos x}{\sin x\cos x\left(\frac{1}{\cos x}-\frac{\sin^2 x}{\cos x}\right)}$$

$$\frac{dy}{dx}=\frac{\sin x+y\cos x}{\sin x(1-\sin^2 x)}$$

$$\frac{dy}{dx}=\sec^2 x + y\,(2\operatorname{cosec} 2x)$$

$$\frac{dy}{dx}+(-2\operatorname{cosec} 2x)\,y=\sec^2 x$$

Compare with $\frac{dy}{dx}+P(x)\,y=Q(x)$

I.F = $e^{\int p(x)dx} = e^{\int -2\operatorname{cosec}2x dx}$

Put 2x = t

dx = $\frac{dt}{2}$

$= e^{-\int 2\operatorname{cosec}t\frac{dt}{2}} = e^{-\int \operatorname{cosec}t\, dt} = e^{-\ln|\tan\frac{t}{2}|}$

$= e^{-\ln|\tan x|} = \frac{1}{|\tan x|}$

Required solution y(IF) = $\int Q(IF)dx + c$

$\frac{y}{|\tan x|} = \int \frac{\sec^2 x}{|\tan x|} dx + c = = \ln|\tan x| + c$

y = |tan x| (ln |tan x| + c) ...(1)

Put x = $\frac{\pi}{4} \Rightarrow y = 2 \Rightarrow 2 = |1|(\ln|1| + c)$

c = 2

put c = 2 in equation (1)

then y = |tan x| (ln|tanx| + 2)

Now $y\left(\frac{\pi}{3}\right) = \sqrt{3}(\ln\sqrt{3} + 2)$

5. The area of the region $\left\{(x,y): y^2 \le 4x, x < 4, \frac{xy(x-1)(x-2)}{(x-3)(x-4)} > 0, x \ne 3\right\}$ is

(1) $\frac{8}{3}$ (2) $\frac{64}{3}$ (3) $\frac{16}{3}$ (4) $\frac{32}{3}$

Sol. (4)

$\{(x, y); y^2 \le 4x, x < 4, \frac{xy(x-1)(x-2)}{(x-3)(x-4)} > 0, x \ne 3\}$

$y^2 \le 4x, x < 4, \frac{xy(x-1)(x-2)}{(x-3)(x-4)} > 0$

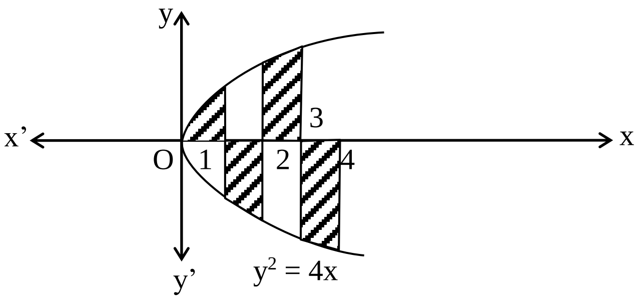
Case (i) $y > 0 \Rightarrow \frac{x(x-1)(x-2)}{(x-3)(x-4)} > 0$

$x \in (0, 1) \cup (2, 3)$

Case (ii) $y < 0 \Rightarrow \frac{x(x-1)(x-2)}{(x-3)(x-4)} < 0$

$x \in (1, 2) \cup (3, 4)$

Area = $\int_0^1 2\sqrt{x}dx + \left|\int_1^2 2\sqrt{x}dx\right| + \int_2^3 2\sqrt{x}dx + \left|\int_3^4 2\sqrt{x}dx\right|$



$$= \frac{4}{3}\left(\left[x^{\frac{3}{2}}\right]_0^1 + \left|\left[x^{\frac{3}{2}}\right]_1^2\right| + \left[x^{\frac{3}{2}}\right]_2^3 + \left|\left(x^{\frac{3}{2}}\right)_3^4\right|\right)$$

$$A = \frac{4}{3}(1 + 2\sqrt{2} - 1 + 3\sqrt{3} - 2\sqrt{2} + 4\sqrt{4} - 3\sqrt{3}) = \frac{4}{3}\times 8 = \frac{32}{3}$$

6. The sum of the series $\frac{1}{1-3\cdot1^2+1^4}+\frac{2}{1-3\cdot2^2+2^4}+\frac{3}{1-3\cdot3^2+3^4}+\ldots$ up to 10 -terms is

(1) $\frac{45}{109}$ (2) $-\frac{55}{109}$ (3) $-\frac{45}{109}$ (4) $\frac{55}{109}$

Sol. **(2)**

$$S = \sum_{r=1}^{10}\frac{r}{1-3r^2+r^4}$$

$$S = \sum_{r=1}^{10}\frac{r}{r^4-2r^2+1-r^2}$$

$$= \sum_{r=1}^{10}\frac{r}{(r^2-1)^2-r^2}$$

$$= \sum_{r=1}^{10}\frac{r}{(r^2-1+r)(r^2-1-r)}$$

$$= \frac{1}{2}\sum_{r=1}^{10}\frac{((r^2-1)+r)-((r^2-1)-r)}{((r^2-1)+r)((r^2-1)-r)}$$

$$= \frac{1}{2}\sum_{r=1}^{10}\left(\frac{1}{r^2-1-r}-\frac{1}{r^2-1+r}\right)$$

$$= \frac{1}{2}\left(\left(-\frac{1}{1}-\frac{1}{1}\right)+\left(\frac{1}{1}-\frac{1}{5}\right)+\ldots+\left(-\frac{1}{109}\right)\right)$$

$$= \frac{1}{2}\left(-1-\frac{1}{109}\right) = \frac{1}{2}\left(\frac{-110}{109}\right)$$

$$S = -\frac{55}{109}$$

7. If the foci of a hyperbola are same as that of the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$ and the eccentricity of the hyperbola is $\frac{15}{8}$ times the eccentricity of the ellipse, then the smaller focal distance of the point $\left(\sqrt{2},\frac{14}{3}\sqrt{\frac{2}{5}}\right)$ on the hyperbola, is equal to

(1) $7\sqrt{\frac{2}{5}}+\frac{8}{3}$ (2) $7\sqrt{\frac{2}{5}}-\frac{8}{3}$ (3) $14\sqrt{\frac{2}{5}}-\frac{4}{3}$ (4) $14\sqrt{\frac{2}{5}}-\frac{16}{3}$

Sol. **(2)**

$\frac{x^2}{9} + \frac{y^2}{25} = 1$

a = 3, b = 5,

$e = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

foci = (0, ± be) = (0, ± 4)

Eccentricity of hyperbola $e_H = \frac{4}{5}\times\frac{15}{8}=\frac{3}{2}$

Let equation of hyperbola be

$\frac{x^2}{A^2} - \frac{y^2}{B^2} = -1$

According to equation $Be_H = 4$

$\therefore B = \frac{8}{3}$

$\because A^2 = B^2 (e_H^2 - 1)$

$= \frac{64}{9}\left(\frac{9}{4}-1\right) \Rightarrow A^2 = \frac{80}{9}$

$\therefore \frac{x^2}{\frac{80}{9}} - \frac{y^2}{\frac{64}{9}} = -1$

Directrix $y = \pm\frac{B}{e_H} = \pm\frac{16}{9}$

PS = ePM

$= \frac{3}{2}\left|\frac{14}{3}\sqrt{\frac{2}{5}}-\frac{16}{9}\right|$

$= 7\sqrt{\frac{2}{5}} - \frac{8}{3}$

8. If the system of linear equations

$x-2y+z=-4$

$2x+\alpha y+3z=5$

$3x-y+\beta z=3$

has infinitely many solutions, then $12\alpha+13\beta$ is equal to

(1) 54 (2) 60 (3) 64 (4) 58

Sol. **(4)**

$x - 2y + z = -4$

$2x + \alpha y + 3z = 5$

$3x - y + \beta z = 3$

For infinite solution $D = 0$, $D_1 = 0$, $D_2 = 0$ & $D_3 = 0$

$$D = 0 \Rightarrow \begin{vmatrix} 1 & -2 & 1 \\ 2 & \alpha & 3 \\ 3 & -1 & \beta \end{vmatrix} = 0$$

$\alpha\beta - 3\alpha + 4\beta = 17$ …(1)

$$D_1 = \begin{vmatrix} -4 & -2 & 1 \\ 5 & \alpha & 3 \\ 3 & -1 & \beta \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & 3 \\ 3 & 3 & \beta \end{vmatrix} = 0$$

$= (5\beta - 9) + 4(2\beta - 9) + 1(6 - 15) = 0$

$13\beta = 54 \Rightarrow \beta = \frac{54}{13}$

Put in (1)

$\frac{54}{13}\alpha - 3\alpha + 4 \times \frac{54}{13} = 17 \Rightarrow \alpha = \frac{1}{3}$

$12\alpha + 13\beta = 12 \times \frac{1}{3} + 13 \times \frac{54}{13} = 58$

9. The solution curve of the differential equation $y\frac{dx}{dy} = x(\log_e x - \log_e y + 1), x > 0, y > 0$ passing through the point (e, 1) is

(1) $\left|\log_e \frac{y}{x}\right| = y^2$ (2) $\left|\log_e \frac{y}{x}\right| = x$ (3) $2\left|\log_e \frac{x}{y}\right| = y+1$ (4) $\left|\log_e \frac{x}{y}\right| = y$

Sol. **(4)**

$$\frac{dx}{dy} = \frac{x}{y}(\log x - \log y + 1)$$

$$\frac{dx}{dy} = \frac{x}{y}\left(\log_e\left(\frac{x}{y}\right)+1\right)$$

Put $\frac{x}{y} = v \Rightarrow \quad x = vy$

$$\frac{dx}{dy} = v + y.\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} = v(\ln v + 1)$$

$$v + y\frac{dv}{dy} = v \ln u + v$$

$$y\frac{dv}{dy} = v \ln v$$

$$\frac{1}{v\ln v}dv = \frac{dy}{y}$$

$$= \int\frac{dv}{v\ln v} = \int\frac{dy}{y}$$

$$\ln|\ln v| = \ln y + \ln c$$

$$\ln|\ln v| = \ln cy$$

$$cy = |\ln v|$$

$$cy = \left|\ln\frac{x}{y}\right| \qquad ...(1)$$

put $x = e$ & $y = 1$

$$c = \left|\ln\frac{e}{1}\right| \Rightarrow c = 1$$

put $c = 1$ in eq. (1)

$$y = \left|\ln\frac{x}{y}\right|$$

10. Three rotten apples are accidently mixed with fifteen good apples. Assuming the random variable x to be the number of rotten apples in a draw of two apples, the variance of x is

(1) $\frac{47}{153}$ (2) $\frac{37}{153}$ (3) $\frac{57}{153}$ (4) $\frac{40}{153}$

Sol. **(4)**

3 rotten apples

15 good apples

Let X is number of rotten apples

$P(X = 0) = \frac{{}^{15}C_2}{{}^{18}C_2} = \frac{105}{153}$

$P(X = 1) = \frac{{}^{3}C_1 \times {}^{15}C_1}{{}^{18}C_2} = \frac{45}{153}$

$P(X = 2) = \frac{{}^{3}C_2}{{}^{18}C_2} = \frac{3}{153}$

$E(X) = 0 \times \frac{105}{153} + 1 \times \frac{45}{153} + 2 \times \frac{3}{153} = \frac{51}{153} = \frac{1}{3}$

$V_{ar}(X) = E(X^2) - (E(X))^2$

$= 0 \times \frac{105}{153}. + 1 \times \frac{45}{153} + 4 \times \frac{3}{153} - \frac{1}{9}$

$= \frac{57}{153} - \frac{1}{9} = \frac{40}{153}$

11. Let a be the sum of all coefficients in the expansion of $\left(1-2x+2x^2\right)^{2023}\left(3-4x^2+2x^3\right)^{2024}$ and $b = \lim_{x\to 0}\left(\frac{\int_0^x \frac{\log(1+t)}{t^{2024}+1}dt}{x^2}\right)$. If the equations $cx^2+dx+e=0$ and $2bx^2+ax+4=0$ have a common root, where $c, d, e \in \mathbb{R}$, then $d:c:e$ equals

(1) 4: 1: 4 (2) 1: 1: 4 (3) 2: 1: 4 (4) 1: 2: 4

Sol. **(2)**

$(1 - 2x + 2x^2)^{2023}. (3 - 4x^2 + 2x^3)^{2024}$

Put x = 1

for getting sum of all coefficient a = 1

$b = \lim_{x\to 0}\left(\frac{\int_0^x \frac{\ln(1+t)}{t^{2024}+1}dt}{x^2}\right)$

$= \lim_{x\to 0}\frac{\frac{\ln(1+x)}{x^{2024}+1}}{2x} = \frac{1}{2}$

$cx^2 + dx + e = 0$

$2bx^2 + ax + 4 = 0$

Have a common root

$cx^2 + dx + e = 0$

$x^2 + x+4 = 0 \rightarrow (D < 0)$

Hence both root are common

$\frac{c}{1} = \frac{d}{1} = \frac{e}{4}$

$c : d : e = 1 : 1 : 4 \Rightarrow d : c : e = 1 : 1 : 4$

12. The distance of the point $Q(0,2,-2)$ from the line passing through the point $P(5,-4,3)$ and perpendicular to the lines $\vec{r}=(-3\hat{i}+2\hat{k})+\lambda(2\hat{i}+3\hat{j}+5\hat{k}),\lambda\in\mathbb{R}$ and $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\mu(-\hat{i}+3\hat{j}+2\hat{k}),\mu\in\mathbb{R}$ is

(1) $\sqrt{54}$ (2) $\sqrt{86}$ (3) $\sqrt{74}$ (4) $\sqrt{20}$

Sol. **(3)**

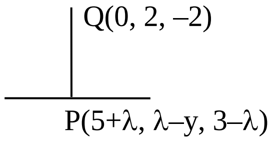
A vector in the direction ratio of the required line
can be obtained by cross product of

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & 3 & 5 \\ -1 & 3 & 2 \end{vmatrix} = -9\,\hat{i} - 9\,\hat{j} + 9\,k$$

Required line $\vec{r} = (5\hat{i} - 4\hat{j} + 3k) + \lambda(-9\hat{i} - 9\hat{j} + 9\hat{k})$

$\vec{r} = (5\hat{i} - 4\,\hat{j} + 3k) + \lambda(\hat{i} + \hat{j} - k)$

Now distance of (0, 2, – 2)



Position vector of point P = $(5+\lambda)\hat{i} + (\lambda-4)\,\hat{j} + (3-\lambda)\hat{k}$

$\overrightarrow{QP} = (5+\lambda)\hat{i} + (\lambda-6)\,\hat{j} + (5-\lambda)\,k$

According to question

$\overrightarrow{QP}\cdot(\hat{i} + \hat{j} - k) = 0$

$\lambda = 2$ Hence point P(7, –2, 1)

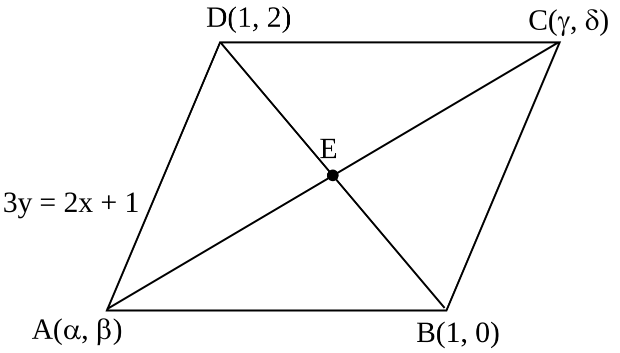
$|\overrightarrow{QP}| = \sqrt{49+16+9} = \sqrt{74}$

13. Let $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$ and let $A(\alpha,\beta), B(1,0), C(\gamma,\delta)$ and D(1, 2) be the vertices of a parallelogram ABCD. If $AB=\sqrt{10}$ and the points A and C lie on the line 3y = 2x + 1, then $2(\alpha+\beta+\gamma+\delta)$ is equal to

(1) 12 (2) 8 (3) 10 (4) 5

Sol. **(2)**

E is mid point at both diagonal



$\left(\frac{\alpha+\gamma}{2}, \frac{\beta+\delta}{2}\right) = (1, 1)$

$\alpha + \gamma = 2$ ...(i)

$\beta + \delta = 2$ ....(ii)

$\alpha + \beta + \gamma + \delta = 4$

$2(\alpha + \beta + \gamma + \delta) = 8$

14. For $\alpha,\beta,\gamma \neq 0$, if $\sin^{-1}\alpha + \sin^{-1}\beta + \sin^{-1}\gamma = \pi$ and $(\alpha+\beta+\gamma)(\alpha-\gamma+\beta) = 3\alpha\beta$, then γ equals

(1) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (2) $\sqrt{3}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{\sqrt{3}}{2}$

Sol. **(4)**

Let $\sin^{-1}\alpha = A$, $\sin^{-1}\beta = B$ & $\sin^{-1}\gamma = C$

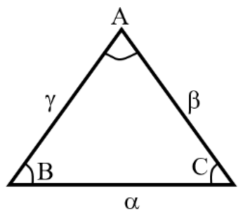
$\alpha = \sin A$, $\beta = \sin B$, $C = \sin\gamma$

$(A + B + C = \pi)$

$\because (\alpha + \beta + \gamma)(\alpha - \gamma + \beta) = 3\alpha\beta$

from sine law

$\frac{\sin A}{\alpha} = \frac{\sin B}{\beta} = \frac{\sin C}{\gamma}$



$(\alpha + \beta)^2 - \gamma^2 = 3\alpha\beta$

$\alpha^2 + \beta^2 - \gamma^2 = \alpha\beta$

$\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta} = \frac{1}{2}$

$\cos C = \frac{1}{2} \Rightarrow \sin C = \frac{\sqrt{3}}{2}$

$\gamma = \sin C = \frac{\sqrt{3}}{2}$

15. Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue and 15 orange marbles, with replacement being made after each drawing. Then the probability, that first drawn marble is red and second drawn marble is white, is

(1) $\frac{2}{25}$ (2) $\frac{4}{75}$ (3) $\frac{4}{25}$ (4) $\frac{2}{3}$

Sol. **(2)**

15 orange
20 blue
30 white
10 red

Probability of drawing first red and second white

(form multiplication theorem)

$$= \frac{{}^{10}C_1}{{}^{75}C_1} \times \frac{{}^{30}C_1}{{}^{75}C_1} = \frac{4}{75}$$

16. Let S be the set of positive integral values of a for which $\frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32} < 0, \forall x \in \mathbb{R}$. Then, the number of elements in S is:

(1) ∞ (2) 3 (3) 0 (4) 1

Sol. **(3)**

$$\frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32} < 0$$

D = 64 – 4 × 32 < 0

& a = 1 > 0

$\therefore$ $x^2 - 8x + 32 > 0 \ \forall \ x \in R$

$ax^2 + 2(a + 1)x + 9a + 4 < 0 \ \forall \ x \in R$

Only possible when

a < 0 & D < 0

but we need positive integral value of a.

So,

No solution

17. Let g(x) be a linear funchion and $f(x) = \begin{cases} g(x) & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{\frac{1}{x}} & , x > 0 \end{cases}$, is continuous at x = 0. If $f'(1) = f(-1)$, then the value $g(3)$ is

(1) $\frac{1}{3}\log_e\left(\frac{4}{9e^{1/3}}\right)$ (2) $\log_e\left(\frac{4}{9e^{1/3}}\right)$ (3) $\frac{1}{3}\log_e\left(\frac{4}{9}\right)+1$ (4) $\log_e\left(\frac{4}{9}\right)-1$

Sol. **(2)**

Let $g(x) = ax + b$

$\because$ function is continuous at $x = 0$

$$\lim_{x \to 0^+} f(x) = f(0) = \lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0^+} \left(\frac{1+x}{2+x}\right)^{\frac{1}{x}} = b$$

b = 0

$\therefore$ g(x) = ax

For x > 0

$$f'(x)=\frac{1}{x}\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}-1}\frac{1}{(2+x)^2}+\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}}\ln\left(\frac{1+x}{2+x}\right)\left(-\frac{1}{x^2}\right)$$

$$f'(1)=\frac{1}{9}-\frac{2}{3}\ln\frac{2}{3}$$

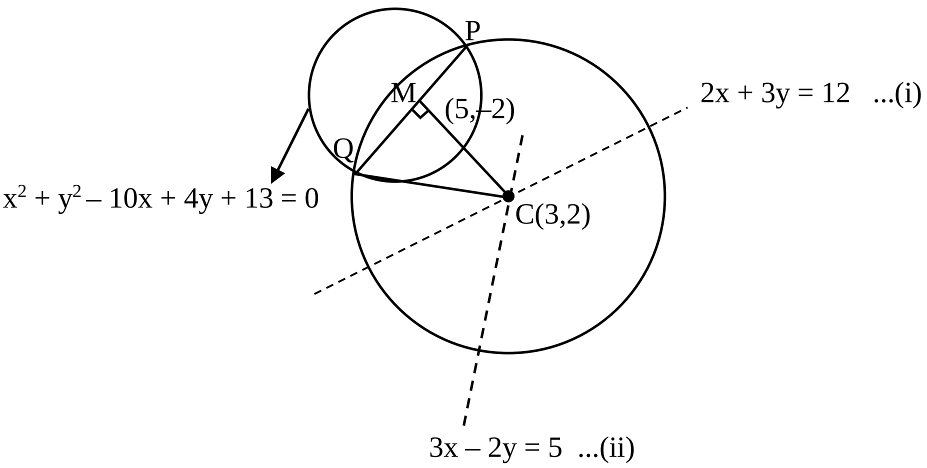
$$f'(1)=f(-1) \Rightarrow \frac{1}{9}-\frac{2}{3}\ln\frac{2}{3}=g(-1)=-a$$

$$a=\frac{2}{3}\ln\frac{2}{3}-\frac{1}{9} \qquad \therefore\ g(3)=2\ln\frac{2}{3}-\frac{1}{3}=\ln\left(\frac{4}{9e^{1/3}}\right)$$

18. If one of the diameters of the circle $x^2+y^2-10x+4y+13=0$ is a chord of another circle C, whose center is the point of intersection of the lines 2x + 3y = 12 and 3x – 2y = 5, then the radius of the circle C is:

(1) $\sqrt{20}$ (2) 4 (3) 6 (4) $3\sqrt{2}$

Sol. **(3)**



Solve (i) and (ii) C(3,2)

Centre of given circle is M(5, –2)

Radius $PM=QM=4$

$CM=\sqrt{4+16}=2\sqrt{5}$

Radius of circle $CP=CQ=\sqrt{16+20}=\sqrt{36}=6$

19. For 0 < c < b < a, let (a + b – 2c) x^2 + (b + c – 2a) x + (c + a – 2b) = 0 and $\alpha \neq 1$ be one of its root. Then, among the two statements

(I) If $\alpha \in (-1, 0)$, then b cannot be the geometric mean of a and c

(II) If $\alpha \in (0, 1)$, then b may be the geometric mean of a and c

(1) Neither (I) nor (II) is true (2) only (II) is true

(3) only (I) is true (4) Both (I) and (II) are true

Sol. (4)

$f(x)=(a+b-2c)x^2+(b+c-2a)x+(c+a-2b)$

$\because f(1)=0$

$\because$ one root is unit

product of roots $\alpha.1=\frac{c+a-2b}{a+b-2c}$

If $\alpha\in(-1,0)\ \Rightarrow\ -1<\frac{c+a-2b}{a+b-2c}<0\ \Rightarrow\ b+c<2a\ \&\ b>\frac{a+c}{2}$

$\therefore$ b cannot be G.M. between a & c

If $\alpha\in(0,1)\ \Rightarrow 0<\frac{c+a-2b}{a+b-2c}<1$

$b>c$ and $b<\frac{a+c}{2}$

$\therefore$ b may be GM between a & c

20. If $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$ and $(fof)(x)=g(x)$, where $g:\mathbb{R}-\left\{\frac{2}{3}\right\}\to\mathbb{R}-\left\{\frac{2}{3}\right\}$, then $(gogog)(4)$ is equal to

(1) 4 (2) – 4 (3) $-\frac{19}{20}$ (4) $\frac{19}{20}$

Sol. (1)

$f(x)=\frac{4x+3}{6x-4}$

$$g(x)=f\{f(x)\}=\frac{4f(x)+3}{6f(x)-4}=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=x$$

$g(x)=x$

Now

$g\{g\{g(x)\}\}=x$

$g(g(g(4)))=4$

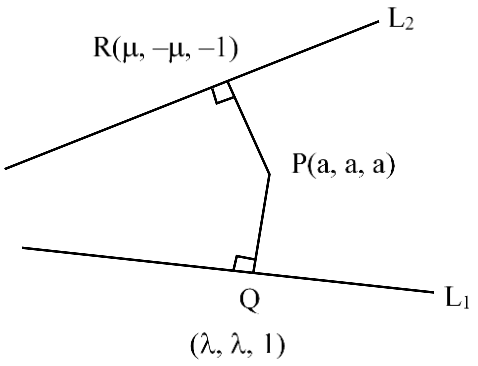
SECTION – B

21. Let Q and R be the feet of perpendiculars from the point $P(a,a,a)$ on the lines x = y, z = 1 and x = – y, z = –1 respectively. If $\angle QPR$ is a right angle, then $12a^2$ is equal to ________

Sol. 12

L_1: $\frac{x}{1}=\frac{y}{1}=\frac{z-1}{0}=\lambda$ (let)

$L_2: \frac{x}{1} = \frac{y}{-1} = \frac{z+1}{0} = \mu(\text{let})$



$PQ \perp L_1$

$(\lambda - a)1 + (\lambda - a)1 + (1 - a)\,0 = 0$

$\lambda = a$

$PR \perp L_2$

$1(\mu - a) + 1\,(\mu + a) + 0\,(-1 - a) = 0$

$\mu - a + \mu + a = 0$

$\mu = 0$

$\therefore$ Point Q (a, a, 1) and R(0, 0, –1)

According to question

$\angle QPR = 90^\circ$

Direction ratio of PR = (a, a, a + 1)

Direction ratio of PQ = (0, 0, a – 1)

$a\times 0 + 0\times a + (a + 1)(a - 1) = 0$

$a^2 = 1$

$\Rightarrow 12a^2 = 12\times 1 = 12$

22. Let $S = (-1, \infty)$ and $f : S \rightarrow \mathbb{R}$ be defined as

$$f(x) = \int_{-1}^{x} \left(e^t - 1\right)^{11} (2t-1)^5 (t-2)^7 (t-3)^{12} (2t-10)^{61} dt,$$

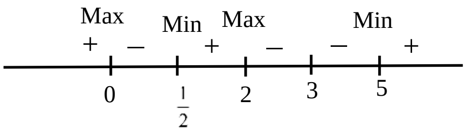
Let $p =$ Sum of squares of the values of x, where f(x) attains local maxima on S, and q = Sum of the values of x, where f(x) attains local minima on S. Then, the value of $p^2 + 2q$ is ________

Sol. **27**

$f(x)=\int_{-1}^{x}\left((e^{t}-1)^{11}(2t-1)^{5}(t-2)^{7}(t-3)^{12}(2t-10)^{61}\right)dt$

Using Lebnitz

$f'(x)=\left(e^{x}-1\right)^{11}(2x-1)^{5}(x-2)^{7}(x-3)^{12}(2x-10)^{61}$



Point of maxima at x = 0, 2

Point of minima at x = $\frac{1}{2}$, 5

$p = 0^2 + 2^2 = 4$

$q = \frac{1}{2}+5=\frac{11}{2}$

$p^2 + 2q = 16 + 11 = 27$

23. Let the foci and length of the latus rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ are $(\pm 5,0)$ and $\sqrt{50}$, respectively. Then, the square of the eccentricity of the hyperbola $\frac{x^2}{b^2}-\frac{y^2}{a^2b^2}=1$ equals

Sol. **51**

Foci $=(\pm 5,0)$, $\frac{2b^2}{a}=\sqrt{50}$

$ae=5$, $\quad b^2=\frac{5\sqrt{2}a}{2}$

$\because\ b^2=a^2\left(1-e^2\right)=\frac{5\sqrt{2}a}{2}$

$a\left(1-e^2\right)=\frac{5\sqrt{2}}{2}$

$\frac{5}{e}\left(1-e^2\right)=\frac{5\sqrt{2}}{2}$

$\sqrt{2}-\sqrt{2}e^2=e$

$\sqrt{2}e^2+e-\sqrt{2}=0$

$\left(e+\sqrt{2}\right)\left(\sqrt{2}e-1\right)=0$

$e=-\sqrt{2},\frac{1}{\sqrt{2}},$

$e=-\sqrt{2}$ (Rejected)

Hyperbola $\frac{x^2}{b^2}-\frac{y^2}{a^2b^2}=1$

$a=5\sqrt{2}$, $b=5$

$(e_H)^2=1+\frac{a^2b^2}{b^2}=1+a^2=51$

24. Let $A=\{1,2,3,4\}$ and $R=\{(1,2),(2,3),(1,4)\}$ be a relation on A. Let S be the equivalence relation on A such that $R\subset S$ and the number of elements in S is n. Then, the minimum value of n is ______

Sol. 16

$A=\{1,2,3,4\}$

$R=\{(1,2)(2,3)(1,4)\}$

For equivalence relation, relation must be reflexive, symmetric & transitive

$R=\{(1,1)(2,2)(3,3)(4,4)(1,2)(2,1)(2,3)$

$(3,2)(1,4)(4,1)(1,3)(3,1)(2,4)(4,2)(4,3)(3,4)\}$

All elements are included

$\therefore$ Answer is 16

25. If the integral $525\int_0^{\frac{\pi}{2}}\sin 2x\cos^{\frac{11}{2}}x\left(1+\cos^{\frac{5}{2}}x\right)^{\frac{1}{2}}dx$ is equal to $(n\sqrt{2}-64)$, then n is equal to ______

Sol. 176

$I=525\int_0^{\pi/2}\sin 2x\cos^{11/2}x\left(1+\cos^{5/2}x\right)^{\frac{1}{2}}dx$

put $\sqrt{\cos x}=t \Rightarrow \cos x=t^2$

– sinx dx = 2tdt

sin x dx = – 2tdt

$I=525\int_0^1 2(2t))t^2t^{11}\sqrt{1+t^5}dt$

$=525\times 4\int_0^1\sqrt{1+t^5}.t^{14}dt$

$=2100\int_0^1\sqrt{1+t^5}.t^{14}dt$

again put $\sqrt{1+t^5}=u \Rightarrow 1+t^5=u^2$

$5t^4dt=2udu$

$I=2100\int_1^{\sqrt{2}}u\cdot\frac{2u}{5}\cdot\left(u^2-1\right)^2du$

$=420\times 2\left(\int_1^{\sqrt{2}}u^2(u^4-2u^2+1)du\right)$

$= 840\int_{1}^{\sqrt{2}}\left(u^6 - 2u^4 + u^2\right)du$

$= 840\left[\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right]_1^{\sqrt{2}}$

$= 840\left[\frac{8\sqrt{2}}{7} - \frac{8\sqrt{2}}{5} + \frac{2\sqrt{2}}{3} - \frac{1}{7} + \frac{2}{5} - \frac{1}{3}\right]$

$= 176\sqrt{2} - 64$

Compare with $n\sqrt{2} - 64$

$n = 176$

26. If α denotes the number of solutions of $|1-i|^x = 2^x$ and $\beta = \left(\frac{|z|}{\arg(z)}\right)$, where $z = \frac{\pi}{4}(1+i)^4\left[\frac{1-\sqrt{\pi}i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}i}\right], i = \sqrt{-1}$, then the distance of the point (α, β) from the line 4x – 3 y = 7 is ______

Sol. **3**

$|1-i|^x = 2^x$ and $\beta = \frac{|z|}{\arg(z)}$

where $z = \frac{\pi}{4}(1+i)^4\left[\frac{1-\sqrt{\pi}i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}i}\right]$

$\because\ |1-i|^x = 2^x \Rightarrow (\sqrt{2})^x = 2^x \Rightarrow x = 0 \Rightarrow \alpha = 1$

$z = \frac{\pi}{4}(1+i)^4\left[\frac{1}{i}\left(\frac{1-\sqrt{\pi}i}{1-\sqrt{\pi}i}\right) + \frac{1}{i}\left(\frac{\sqrt{\pi}-i}{\sqrt{\pi}-i}\right)\right]$

$= \frac{\pi}{4}(1+i)^4\left(\frac{2}{i}\right)$

$= \frac{\pi}{2}\left((1+i)^2\right)^2\frac{1}{i} = \frac{\pi}{2}(2i)^2\frac{1}{i}$

$= \frac{\pi}{2}\frac{\left(4i^2\right)}{i}$

$z = 2\pi i$

$\beta = \frac{|z|}{\arg(z)} = \frac{2\pi}{\pi/2} = 4$

distance from point $(1,4)$ to the line $4x - 3y = 7$

$= \frac{|4\times1 - 3\times4 - 7|}{\sqrt{9+16}} = \frac{15}{5} = 3$

27. The total number of words (with or without meaning) that can be formed out of the letters of the word 'DISTRIBUTION' taken four at a time, is equal to

Sol. **3734**

DISTRIBUTION

Total letters = 12	I I I 3		S 1	U 1
	T T 2	R 1	O 1	
	D 1	B 1	N 1	

Case I: 3 alike and 1 different $= {}^1C_1 \times {}^8C_1 \times \frac{4!}{3!} = 32$

Case II: 2 alike and 2 another alike $= {}^2C_2 \times \frac{4!}{2!2!} = 6$

Case III: 2 alike and 2 different $= {}^2C_1 \times {}^8C_2 \times \frac{4!}{2!} = 672$

Case IV: 4 different $= {}^9C_4 \times 4! = 3024$

Then total number of words can be formed

= 32 + 6 + 672 + 3024 = 3734

28. In the expansion $(1+x)(1-x^2)\left(1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}\right)^5, x \neq 0$, then sum of the coefficients of x^3 and x^{-13} is equal to ______

Sol. **118**

$(1 + x)(1 - x^2)\left(1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}\right)^5$

$(1 + x)(1 - x^2)\left(1+\frac{1}{x}\right)^{15}$

$\frac{(1+x)(1-x^2)(1+x)^{15}}{x^{15}} = \frac{(1-x)(1+x)^{17}}{x^{15}}$

$= \frac{(1+x)^{17} - x(1+x)^{17}}{x^{15}}$

Coefficient of $x^3 = 0 - 1 = -1$

Coefficient of x^{-13} = Coefficient of $x^2 = (1 + x)^{17} - x(1 + x)^{17}$

$= {}^{17}C_2 - {}^{17}C_1$

= 17 × 8 – 17 = 119

Sum of coefficient of x^3 and $x^{-13} = -1 + 119 = 118$

29. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}|=1, |\vec{b}|=4$, and $\vec{a}\cdot\vec{b}=2$. If $\vec{c}=(2\vec{a}\times\vec{b})-3\vec{b}$ and the angle between $\vec{b}$ and $\vec{c}$ is α, then $192\sin^2\alpha$ is equal to ______

Sol. **48**

$|\vec{a}| = 1, |\vec{b}| = 4$ and $\vec{a} . \vec{b} = 2$

$\vec{b} . \vec{c} = (2\vec{a} \times \vec{b}) . \vec{b} - 3|\vec{b}|^2$

$|\vec{b}||\vec{c}| \cos\alpha = -3|\vec{b}|^2$

$|c| \cos\alpha = -12$ …(1)

$\because |\vec{b}| = 4$

$\vec{a} . \vec{b} = 2$

$|\vec{a}||\vec{b}| \cos\theta = 2$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$|\vec{c}|^2 = |(2\vec{a} \times \vec{b}) - 3\vec{b}|^2$

$= |2\vec{a} \times \vec{b}|^2 + 9|\vec{b}|^2 - 6(2\vec{a} \times \vec{b}).\vec{b}$

$= 4|a|^2|\vec{b}|^2 \sin^2\theta + 9 \times 16 - 0$

$= 4 \times 1 \times 16 \times \frac{3}{4} + 144$

$= 48 + 144$

$|\vec{c}|^2 = 192$

From (1)

$|c|^2 \cos^2\alpha = 144$

$192 \cos^2\alpha = 144$

$192 (1 - \sin^2\alpha) = 144$

$192 - 192 \sin^2\alpha = 144$

$192 \sin^2\alpha = 48$

30. Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a function defined by $f(x)=\frac{4^x}{4^x+2}$ and $M=\int_{f(a)}^{f(1-a)} x\sin^4(x(1-x))dx$, $N=\int_{f(a)}^{f(1-a)} \sin^4(x(1-x))dx; a\neq\frac{1}{2}$. If $\alpha M=\beta N, \alpha,\beta\in\mathbb{N}$, then the least value of $\alpha^2+\beta^2$ is equal to ______.

Sol. **5**

$\because f(x)=\dfrac{4^x}{4^x+2}$

$f(1-x)=\dfrac{4^{1-x}}{4^{1-x}+2}$

$f(x)+f(1-x)=1$

$\therefore$ f(a) + f(1 – a) = 1

$M=\int_{f(a)}^{f(1-a)} x\sin^4(x(1-x))dx$

$M=\dfrac{1}{2}\int_{f(a)}^{f(1-a)} \sin^4(x(1-x))dx$ (Using elimination of x)

$M=\dfrac{N}{2}\Rightarrow 2M=N$

$\alpha M=\beta N$

$\alpha=2\ \&\ \beta=1$

$\alpha^2+\beta^2=4+1=5$

REVISED