JEE MAIN 2023
Paper with Solution

MATHS | 29th Jan 2023 _ Shift-1

Umeed Rank Ki Ho Ya Selection Ki, JEET NISCHIT HAI!

Most Promising RANKS
Produced by MOTION Faculties

NEET / AIIMS

- **AIR-1 to 10** — 25 Times
- **AIR-11 to 50** — 83 Times
- **AIR-51 to 100** — 81 Times

JEE MAIN+ADVANCED

- **AIR-1 to 10** — 8 Times
- **AIR-11 to 50** — 32 Times
- **AIR-51 to 100** — 36 Times

Nation's Best SELECTION
Percentage (%) Ratio

Student Qualified in NEET

(2022)
4837/5356 = **90.31%**
(2021)
3276/3411 = **93.12%**

Student Qualified in JEE ADVANCED

(2022)
1756/4818 = **36.45%**
(2021)
1256/2994 = **41.95%**

Student Qualified in JEE MAIN

(2022)
4818/6653 = **72.41%**
(2021)
2994/4087 = **73.25%**

NITIN VIIJAY (NV Sir)
Founder & CEO

Section A

61. Let α and β be real numbers. Consider a 3×3 matrix A such that $A^2 = 3A + \alpha I$. If $A^4 = 21A + \beta I$, then

(1) $\beta = -8$ (2) $\beta = 8$ (3) $\alpha = 4$ (4) $\alpha = 1$

Sol. **1**

$A^2 = 3A + \alpha I$ …… (1)

and $A^4 = 21A + \beta I$ ……..(2)

Now $A^4 = A^2 . A^2$

$A^4 = (3A + \alpha I) . (3A + \alpha I)$ {from (1)}

$A^4 = 9A^2 + 6\alpha A + \alpha^2 I$ ……..(3)

From (2) and (3)

$9A^2 + 6\alpha A + \alpha^2 I = 21 A + \beta I$

putting value of A^2 from (1)

$9(3A + \alpha I) + 6\alpha A + \alpha^2 I = 21 A + \beta I$

$(27 + 6\alpha)A + (9\alpha + \alpha^2) I = 21A + \beta I$

by comparison

$27 + 6\alpha = 21$ and $9\alpha + \alpha^2 = \beta$

$\Rightarrow 6\alpha = -6$ putting $\alpha = -1$

$\Rightarrow \alpha = -1$ $\therefore \beta = -8$

62. Let $x = 2$ be a root of the equation $x^2 + px + q = 0$ and

$$f(x) = \begin{cases} \dfrac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x - 2p)^4} & , x \neq 2p \\ 0, & , x = 2p \end{cases}$$

$\lim_{x \to 2p^+} [f(x)]$

where $[\cdot]$ denotes greatest integer function, is

(1) 0 (2) -1 (3) 2 (4) 1

Sol. **1**

$$f(x) = \begin{cases} \dfrac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x - 2p)^4} & , x \neq 2p \\ 0 & , \quad x = 2p \end{cases}$$

$\because$ x = 2 is a root of equation $x^2 + px + q = 0$

$\therefore 4 + 2p + q = 0$

$\Rightarrow 2p = -q - 4$

$\Rightarrow 4p^2 = (q + 4)^2 = q^2 + 8q + 16$ …….(1)

Now $\lim\limits_{x \to 2p^+} f(x) = \lim\limits_{x \to 2p^+} \dfrac{1 - \cos(x^2 - 4px + 4p^2)}{(x - 2p)^4}$ (from (1))

$$= \lim_{x \to 2p^+} \left[\frac{1 - \cos(x - 2p)^2}{\{(x - 2p)^2\}^2} \right]$$

$$= \frac{1}{2} \quad \left\{ \because \lim_{x \to 0} \frac{1 - \cos\theta}{\theta^2} = \frac{1}{2} \right\}$$

$$\therefore \lim_{x \to 2p^+} [f(x)] = \left[\frac{1}{2} \right] = 0$$

63. Let B and C be the two points on the line $y+x=0$ such that B and C are symmetric with respect to the origin. Suppose A is a point on $y-2x=2$ such that $\triangle ABC$ is an equilateral triangle. Then, the area of the $\triangle ABC$ is

(1) $\frac{10}{\sqrt{3}}$ (2) $3\sqrt{3}$ (3) $2\sqrt{3}$ (4) $\frac{8}{\sqrt{3}}$

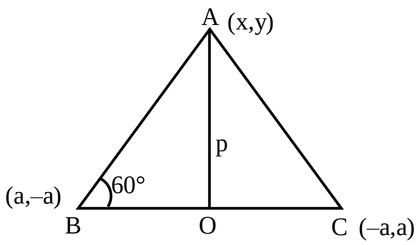
Sol. **4**

Since, A lies on perpendicular bisector of BC, whose equation is

y = x ……..(1)

Now, A is the point of intersection of y = x and y – 2x = 2

$\therefore$ point A, after solving is A(–2, –2)



In ΔAOC $\tan 60° = \frac{p}{OC} \Rightarrow OC = \frac{p}{\sqrt{3}}$ $\{\because OA = p\}$

$\therefore BC = 2 \times OC = \frac{2p}{\sqrt{3}}$

Now, Area of $\Delta ABC = \frac{1}{2} \times BC \times OA$

$= \frac{1}{2} \times \frac{2p}{\sqrt{3}} \times p = \frac{p^2}{\sqrt{3}}$ sq. unit

and $p = OA = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

So, Area of $\Delta ABC = \frac{(2\sqrt{2})^2}{\sqrt{3}} = \frac{8}{\sqrt{3}}$ sq. unit

64. Consider the following system of equations

$\alpha x + 2y + z = 1$

$2\alpha x + 3y + z = 1$

$3x + \alpha y + 2z = \beta$

for some $\alpha, \beta \in \mathbb{R}$. Then which of the following is NOT correct.

(1) It has a solution if $\alpha = -1$ and $\beta \neq 2$

(2) It has a solution for all $\alpha \neq -1$ and $\beta = 2$

(3) It has no solution for $\alpha = 3$ and for all $\beta \neq 2$

(4) It has no solution for $\alpha = -1$ and for all $\beta \in \mathbb{R}$

Sol. **4**

$$\because D = \begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & 2 \end{vmatrix}$$

$D = \alpha(6-\alpha) + 2(3-4\alpha) + 1(2\alpha^2 - 9)$

$= 6\alpha - \alpha^2 + 6 - 8\alpha + 2\alpha^2 - 9$

$D = \alpha^2 - 2\alpha - 3$

for no solution , D = 0

$\Rightarrow \quad \alpha^2 - 2\alpha - 3 = 0$

$(\alpha + 1)(\alpha - 3) = 0$

$\Rightarrow \quad \alpha = -1, \alpha = 3$

Now,

$$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ \beta & \alpha & 2 \end{vmatrix}, D_2 = \begin{vmatrix} \alpha & 1 & 1 \\ 2\alpha & 1 & 1 \\ 3 & \beta & 2 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & \beta \end{vmatrix}$$

if $\alpha = -1$ then

$$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ \beta & -1 & 2 \end{vmatrix}, D_2 = \begin{vmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 3 & \beta & 2 \end{vmatrix}, D_3 = \begin{vmatrix} -1 & 2 & 1 \\ -2 & 3 & 1 \\ 3 & -1 & \beta \end{vmatrix}$$

$\Rightarrow$ only for $\beta = 2$, $D_1 = 0$, $D_2 = 0$, $D_3 = 0$

$\therefore$ It has no solution if $\alpha = -1$ and $\beta \neq 2$

if $\alpha = 3$

$$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ \beta & 3 & 2 \end{vmatrix}, D_2 = \begin{vmatrix} 3 & 1 & 1 \\ 6 & 1 & 1 \\ 3 & \beta & 2 \end{vmatrix}, D_3 = \begin{vmatrix} 3 & 2 & 1 \\ 6 & 3 & 1 \\ 3 & 3 & \beta \end{vmatrix}$$

$\Rightarrow$ Only for $\beta = 2$, $D_1 = D_2 = D_3 = 0$

$\Rightarrow$ It has no solution for $\beta \neq 2$

$\therefore$ It has no solution for $\alpha = 3$ and for all $\beta \neq 2$

65. Let $y = f(x)$ be the solution of the differential equation $y(x+1)dx - x^2dy = 0, y(1) = e$. Then $\lim_{x\to 0^+} f(x)$ is equal to

(1) $\frac{1}{e^2}$ (2) e^2 (3) 0 (4) $\frac{1}{e}$

Sol. **3**

$y(x+1)dx - x^2 dy = 0, \qquad y(1) = e$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x+1)}{x^2}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{(x+1)dx}{x^2}$$

$$\ell ny = \ell nx - \frac{1}{x} + c$$

$\because y(1) = e$

$\therefore 1 = 0 - 1 + C \Rightarrow C = 2$

$$\text{Now, } \ell ny = \ell nx - \frac{1}{x} + 2$$

$\Rightarrow \ell n\left(\frac{y}{x}\right)=2-\frac{1}{x}$

$\Rightarrow \frac{y}{x}=e^{2-\frac{1}{x}}$

$\Rightarrow y = x,\ e^{2-\frac{1}{x}}$

So, $\lim_{x\to 0^+} y = \lim_{x\to 0^+} x e^{2-\frac{1}{x}} = 0$

66. The domain of $f(x) = \frac{\log_{(x+1)}(x-2)}{e^{2\log_e x}-(2x+3)}, x \in \mathbb{R}$ is

(1) $\mathbb{R}-\{3\}$ (2) $(-1,\infty)-\{3\}$ (3) $(2,\infty)-\{3\}$ (4) $\mathbb{R}-\{-1,3\}$

Sol. **3**

$f(x) = \frac{\log_{(x+1)}(x-2)}{e^{2\ln x}-(2x+3)}$

case (i) $x - 2 > 0 \Rightarrow x > 2$

$x \in (2, \infty)$

case (ii) $x + 1 > 0$ and $x + 1 \neq 1$

$x > -1$, $x \neq 0$

$\therefore\ x_t(-1,0)\cup(0,\infty)$

case (iii) $x > 0 \Rightarrow x_t\ (0, \infty)$

case (iv) $e^{2\ell nx} - (2x + 3) \neq 0$

$\Rightarrow$ $x^2 - 2x + 3 \neq 0$

$(x - 3)(x + 1) \neq 0$

$\Rightarrow$ $x \neq 3 , x \neq -1$

$\therefore$ from (i) n (ii) n (iii)n (iv)

$x_t\ (2, \infty) - \{3\}$

67. Fifteen football players of a club-team are given 15 T-shirts with their names written on the backside. If the players pick up the T-shirts randomly, then the probability that at least 3 players pick the correct T-shirt is

(1) $\frac{5}{24}$ (2) $\frac{1}{6}$ (3) $\frac{5}{36}$ (4) $\frac{2}{15}$

Sol. **2**

Required probability = $1 - \frac{D_{(15)} + {}^{15}C_1 D_{(14)} + {}^{15}C_2 D_{(3)}}{15!}$

Taking $D_{(15)}$ as $\frac{15!}{e}$

$D_{(14)}$ as $\frac{14!}{e}$

$D_{(13)}$ as $\frac{13!}{e}$

We get $1-\left(\dfrac{\frac{15!}{e}+15\frac{14!}{e}+\frac{15\times 14}{2\times 1}\times\frac{13!}{e}}{15!}\right)$

$= 1-\left(\frac{1}{e}+\frac{1}{e}+\frac{1}{2e}\right)=1-\frac{5}{2e}\simeq 0.08$

68. Let $[x]$ denote the greatest integer $\leq x$. Consider the function $f(x)=\max\{x^2,1+[x]\}$. Then the value of the integral $\int_0^2 f(x)dx$ is

(1) $\frac{5+4\sqrt{2}}{3}$ (2) $\frac{4+5\sqrt{2}}{3}$ (3) $\frac{1+5\sqrt{2}}{3}$ (4) $\frac{8+4\sqrt{2}}{3}$

Sol. **1**

f(x) = Max. $\{x^2, 1+[x]\}$

Now, $f(x)=\begin{cases}1+[x] & 0\leq x\leq\sqrt{2}\\ x^2 & \sqrt{2}<x\leq 2\end{cases}$

$\int_0^2 f(x)dx=\int_0^{\sqrt{2}}(1+[x])dx+\int_{\sqrt{2}}^2 x^2dx$

$=\int_0^1 1dx+\int_1^{\sqrt{2}}2dx+\int_{\sqrt{2}}^2 x^2dx$

$=(x)_0^1+2(x)_1^{\sqrt{2}}+\frac{1}{3}(x^3)_{\sqrt{2}}^2$

$=1+2(\sqrt{2}-1)+\frac{1}{3}(8-2\sqrt{2})$

$=\frac{4\sqrt{2}+5}{3}$

69. For two non-zero complex numbers z_1 and z_2, if $\operatorname{Re}(z_1z_2)=0$ and $\operatorname{Re}(z_1+z_2)=0$, then which of the following are possible?

A. $\operatorname{Im}(z_1)>0$ and $\operatorname{Im}(z_2)>0$
B. $\operatorname{Im}(z_1)<0$ and $\operatorname{Im}(z_2)>0$
C. $\operatorname{Im}(z_1)>0$ and $\operatorname{Im}(z_2)<0$
D. $\operatorname{Im}(z_1)<0$ and $\operatorname{Im}(z_2)<0$

Choose the correct answer from the options given below:

(1) B and D (2) A and B (3) B and C (4) A and C

Sol. **3**

$\operatorname{Re}(z_1z_2)=0$ and $\operatorname{Re}(z_1+z_2)=0$

Let $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$

$z_1z_2=(a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2)$

$\because\ \operatorname{Re}(z_1z_2)=a_1a_2-b_1b_2=0$

$\therefore\ a_1a_2=b_1b_2$ ....(1)

and $\operatorname{Re}(z_1+z_2)=0\Rightarrow a_1+a_2=0$

$\Rightarrow a_2 = -a_1$ ....(2)

from (1) and (2)

$b_1 b_2 = -a_1^2 < 0$

Product of $b_1 b_2$ is Negative.

$\therefore$ $\text{Im}(z_1)$ and $\text{Im}(z_2)$ are also of opposite sign.

70. If the vectors $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}, \vec{b} = 2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$ are coplanar and the projection of $\vec{a}$ on the vector $\vec{b}$ is $\sqrt{54}$ units, then the sum of all possible values of $\lambda + \mu$ is equal to

(1) 0 (2) 24 (3) 6 (4) 18

Sol. **2**

Vector $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$

are coplanar then

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0 \qquad \Rightarrow \begin{vmatrix} \lambda & \mu & 4 \\ -2 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$\Rightarrow 10\lambda - 2\mu - 56 = 0$

$\Rightarrow 5\lambda - \mu = 28$ ....(1)

also projection of $\vec{a}$ on the $\vec{b}$ is $\sqrt{54}$ units. then

$\vec{a}.\vec{b} = \sqrt{54}$

$$\Rightarrow \frac{-2\lambda + 4\mu - 8}{\sqrt{24}} = \sqrt{54}$$

$\Rightarrow -2\lambda + 4\mu - 8 = 36$

$\Rightarrow -2\lambda + 4\mu = 44$ ....(2)

from (1) and (2)

$$\lambda = \frac{26}{3} \text{ and } \mu = \frac{46}{3}$$

$$\Rightarrow \lambda + \mu = \frac{26+46}{3} = \frac{72}{3} = 24$$

71. Let $f(\theta) = 3\left(\sin^4\left(\frac{3\pi}{2} - \theta\right) + \sin^4(3\pi + \theta)\right) - 2(1 - \sin^2 2\theta)$ and $S = \left\{\theta \in [0, \pi] : f'(\theta) = -\frac{\sqrt{3}}{2}\right\}$. If $4\beta = \sum_{\theta \in S} \theta$, then $f(\beta)$ is equal to

(1) $\frac{5}{4}$ (2) $\frac{3}{2}$ (3) $\frac{9}{8}$ (4) $\frac{11}{8}$

Sol. **2**

$$f(\theta) = 3\left(\sin^4\left(\frac{3\pi}{2} - \theta\right) + \sin^4(3\pi + \theta)\right) - 2(1 - \sin^2 2\theta)$$

$$= 3\left(\cos^4\theta + \sin^4\theta\right) - 2\cos^2 2\theta$$

$$= 3\left(1 - \frac{\sin^2 2\theta}{2}\right) - 2\cos^2 2\theta$$

$=3\left(\frac{2-\sin^2 2\theta}{2}\right)-2\cos^2 2\theta$

$=3\left(\frac{1+\cos^2 2\theta}{2}\right)-2\cos^2 2\theta$

$f(\theta) = \frac{3-\cos^2 2\theta}{2}$

$f^1(\theta) = \frac{2}{2}\cos 2\theta \sin 2\theta \times 2$

$f^1(\theta) = \sin 4\theta = \frac{-\sqrt{3}}{2}$

$\theta \in [0, \pi]$

$4\theta \in [0, 4\pi]$

$\sin 4\theta = \frac{-\sqrt{3}}{2}$

$4\theta = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{6}, \frac{11\pi}{12}$

$4\beta = \sum_{\theta \in s} \theta = \frac{\pi}{3} + \frac{5\pi}{12} + \frac{5\pi}{6} + \frac{11\pi}{12} = \frac{4\pi + 5\pi + 10\pi + 11\pi}{12} = \frac{30\pi}{12} = \frac{5\pi}{2}$

$\beta = \frac{5\pi}{8}$

$f(\beta) = f\left(\frac{5\pi}{8}\right) = \frac{3-\cos^2\left(\frac{5\pi}{4}\right)}{2} = \frac{3-\frac{1}{2}}{2} = \frac{5}{4}$

72. If p, q and r three propositions, then which of the following combination of truth values of p, q and r makes the logical expression $\{(p \vee q) \wedge ((\sim p) \vee r)\} \rightarrow ((\sim q) \vee r)$ false?

(1) p = T, q = T, r = F (2) p = T, q = F, r = T

(3) p = F, q = T, r = F (4) p = T, q = F, r = F

Sol. **3**

$(p \vee q) \vee (\sim p) \vee r) \rightarrow ((\sim q) \vee r)$

$T \rightarrow F \equiv F$

$\therefore (p \vee q) \wedge ((\sim p) \vee r) \equiv T$ ....(1)

$(\sim q) \vee r \equiv F$ ....(2)

$\Rightarrow \sim q = F, r = F$

$\Rightarrow q = T$

From (1) $p \vee q \equiv T$

$\sim p \vee r \equiv T$

$\therefore r = F$

$\Rightarrow \sim p = T$

$\Rightarrow p = F$

$\therefore$ p = F, q = T, r = F

73. Let Δ be the area of the region $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 21, y^2 \le 4x, x \ge 1\}$. Then $\frac{1}{2}\left(\Delta - 21\sin^{-1}\frac{2}{\sqrt{7}}\right)$ is equal to

(1) $2\sqrt{3} - \frac{2}{3}$ (2) $\sqrt{3} - \frac{4}{3}$ (3) $\sqrt{3} - \frac{2}{3}$ (4) $2\sqrt{3} - \frac{1}{3}$

Sol. 2

Area of Required Region

$$\Delta = 2\left[\int_1^3 2\sqrt{x}\,dx + \int_3^{\sqrt{21}} \sqrt{21 - x^2}\,dx\right]$$

$$= 2\left[2\frac{\left(x^{3/2}\right)_1^3}{(3/2)} + \left\{\frac{(21)}{2}\sin^{-1}\left(\frac{x}{\sqrt{21}}\right) + \frac{x}{2}\sqrt{21 - x^2}\right\}_3^{\sqrt{21}}\right]$$

$$= 2\left[4\sqrt{3} - \frac{4}{3}\right] + (21\sin^{-1}1 + 0) - \left(21\sin^{-1}\left(\frac{3}{\sqrt{21}}\right) + 3\sqrt{12}\right)$$

$$\Delta = 8\sqrt{3} - \frac{8}{3} + \frac{21\pi}{2} - 6\sqrt{3} - 21\sin^{-1}\sqrt{\frac{3}{7}}$$

$$\Delta = 2\sqrt{3} + \frac{21\pi}{2} - \frac{8}{3} - 21\sin^{-1}\left(\sqrt{\frac{3}{7}}\right)$$

Now,

$$\frac{1}{2}\left(\Delta_1 - 21\sin^{-1}\frac{2}{\sqrt{7}}\right) = \frac{1}{2}\left[2\sqrt{3} + \frac{21}{2}\pi - \frac{8}{3} - 21\sin^{-1}\left(\sqrt{\frac{3}{7}}\right) - 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right]$$

$$= \frac{1}{2}\left[2\sqrt{3} + \frac{21}{2}\pi - \frac{8}{3} - 21\sin^{-1}1\right]$$

$$\left\{\text{using} \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right\}\right\}$$

$$= \frac{1}{2}\left[2\sqrt{3} - \frac{8}{3}\right] = \sqrt{3} - \frac{4}{3}$$

74. A light ray emits from the origin making an angle 30° with the positive x-axis. After getting reflected by the line $x + y = 1$, if this ray intersects x-axis at Q, then the abscissa of Q is

(1) $\frac{\sqrt{3}}{2(\sqrt{3}+1)}$ (2) $\frac{2}{3+\sqrt{3}}$ (3) $\frac{2}{(\sqrt{3}-1)}$ (4) $\frac{2}{3-\sqrt{3}}$

Sol. 2

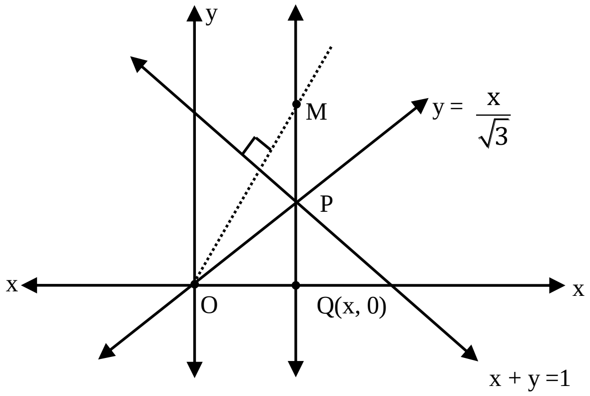
Equation of ray is

$$y = \frac{1}{\sqrt{3}}x \qquad \text{........(1)}$$

Image of 0(0, 0) in the line x + y = 1 is lies on reflected ray.

$$\frac{x - 0}{1} = \frac{y - 0}{1} = -2\frac{(0 + 0 - 1)}{2}$$

$\Rightarrow$ M (1, 1)



$\therefore$ Point of Intersection of lines $y = \frac{x}{\sqrt{3}}$ and x + y = 1 is p (x, y)

$\therefore p\left(\frac{3-\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}\right)$

Now Reflected Ray is same as line passing through PM.

$\therefore$ Slope of PM = $\frac{\frac{\sqrt{3}-1}{2}-1}{\frac{3-\sqrt{3}}{2}-1} = \frac{\sqrt{3}-3}{1-\sqrt{3}} = \sqrt{3}$

Equation of PM whose slope is $\sqrt{3}$ and passing through M (1, 1).

$y - 1 = \sqrt{3}(x - 1)$

$y = \sqrt{3}\ x + \left(-\sqrt{3}+1\right)$

$\because$ ray, Intersects x-axis at $\alpha(x, 0)$

$\therefore$ y = 0

$\Rightarrow \sqrt{3}x = -1\left(-\sqrt{3}+1\right) \Rightarrow \sqrt{3}x = \sqrt{3}-1$

$\Rightarrow x = 1 - \frac{1}{\sqrt{3}}$

$x = \frac{\sqrt{3}-1}{\sqrt{3}} \times \frac{\sqrt{3}+1}{\left(\sqrt{3}+1\right)} = \frac{2}{3+\sqrt{3}}$

$\therefore$ abscissa of α is $\frac{2}{3+\sqrt{3}}$

75. Let $A = \{(x, y) \in \mathbb{R}^2 : y \geq 0, 2x \leq y \leq \sqrt{4-(x-1)^2}\}$ and

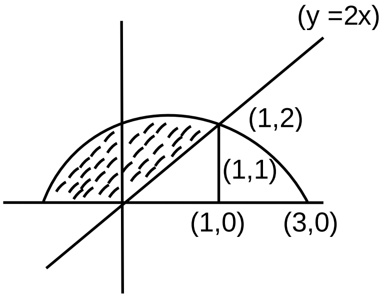
$$B = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq y \leq \min\left\{2x, \sqrt{4-(x-1)^2}\right\}\right\}.$$

Then the ratio of the area of A to the area of B is

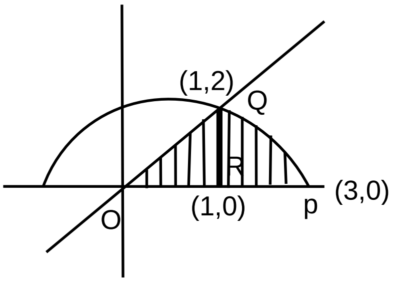
(1) $\frac{\pi+1}{\pi-1}$ (2) $\frac{\pi}{\pi-1}$ (3) $\frac{\pi-1}{\pi+1}$ (4) $\frac{\pi}{\pi+1}$

Sol. **3**

A =



B =



$y^2 = 4 - (x - 1)^2$

$(x - 1)^2 + y^2 = 2^2$

$y = 2x$

$(x - 1)^2 + 4x^2 = 4$

$x^2 + 1 - 2x + 4x^2 = 4$

$5x^2 - 2x - 3 = 0$

$5x^2 - 5x + 3x - 3 = 0$

$5x(x - 1) + 3(x - 1) = 0$

$x = 1, -3/5$

For B : req. area = ar (ΔDRQ) + ar (RPQ)

$$= \frac{1}{2}\times 1\times 2 + \int_1^3 \sqrt{4-(x-1)^2}\,dx$$

$$= 1 + \left[\left(\frac{x-1}{2}\right)\sqrt{4-(x-1)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-1}{2}\right)\right]_1^3$$

$= 1 + 2\sin^{-1} 1 = 1 + \pi$ ………(1)

For A : req. area = area of semi circle – shaded area of B

$$= \frac{\pi r^2}{2} - (1+\pi)$$

$$= \frac{\pi\times 4}{2} - (1+\pi) \qquad \{\because r = 2\}$$

$A = \pi - 1$ …….. (2)

$$\therefore \frac{A}{B} = \frac{\pi - 1}{\pi + 1}$$

76. Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then $\frac{3\alpha}{\beta}$ and $\frac{4\alpha}{\gamma}$ are the roots of the equation

(1) $49x^2 - 245x + 250 = 0$
(2) $7x^2 + 245x - 250 = 0$
(3) $7x^2 - 245x + 250 = 0$
(4) $49x^2 + 245x + 250 = 0$

Sol. **1**

$14x^2 - 31x + 3\lambda = 0$ → roots α, β

and $35x^2 - 53x + 4\lambda = 0$ → roots α, γ

Now, one root is common then

$\therefore\ 14\alpha^2 - 31\alpha + 3\lambda = 0$ .... (1)

$35\alpha^2 - 53\alpha + 4\lambda = 0$ .... (2)

$$\frac{\alpha^2}{-124\lambda + 159\lambda} = \frac{-\alpha}{56\lambda - 105\lambda} = \frac{1}{343}$$

$$\Rightarrow \frac{\alpha^2}{35\lambda} = \frac{\alpha}{49\lambda} = \frac{1}{343}$$

$\Rightarrow \alpha = \frac{\lambda}{7}$ {from (ii) and (iii)}

and $\alpha^2 = \frac{35\lambda}{343}$

$$\Rightarrow \frac{\lambda^2}{49} = \frac{35\lambda}{343}$$

$\lambda^2 - 5\lambda = 0$

$\lambda(\lambda - 5) = 0$

$\lambda = 0, \lambda = 5 \quad \Rightarrow \alpha = 5/7$

not possible $\therefore$ only $\lambda = 5$ possible

Now, $\alpha + \beta = \frac{31}{14}$, $\alpha\beta = \frac{3\lambda}{14}$, $\alpha + \gamma = \frac{53}{35}$, $\alpha\gamma = \frac{4\lambda}{35}$

$\therefore\ \beta = \frac{3}{2}$ and $\gamma = \frac{4}{5}$

Now equation having roots $\left(\frac{3\alpha}{\beta}, \frac{4\alpha}{\gamma}\right) = \left(\frac{10}{7}, \frac{25}{7}\right)$ is

$$x^2 - \frac{35}{7}x + \frac{250}{49} = 0$$

$\Rightarrow 49x^2 - 245x + 250 = 0$

77. Let the tangents at the points $A(4, -11)$ and $B(8, -5)$ on the circle $x^2 + y^2 - 3x + 10y - 15 = 0$, intersect at the point C. Then the radius of the circle, whose centre is C and the line joining A and B is its tangent, is equal to

(1) $2\sqrt{13}$
(2) $\sqrt{13}$
(3) $\frac{3\sqrt{3}}{4}$
(4) $\frac{2\sqrt{13}}{3}$

Sol. **4**

Equation of line AB is

$$y + 5 = \left(\frac{-5+11}{8-4}\right)(x-8)$$

$\Rightarrow y + 5 = \frac{3}{2}(x - 8)$ Þ $2y + 10 = 3x - 24$

$3x - 2y - 34 = 0$ ....(i)

Let C be (h, k) then equation of AB

$$hx + ky - \frac{3}{2}(x + h) + 5(y + k) - 15 = 0$$

$$x\left(h - \frac{3}{2}\right) + y(k + 5) - \frac{3}{2}h + 5k - 15 = 0 \qquad \text{........(ii)}$$

Now, by comparing (i) and (ii)

$$\frac{h-\frac{3}{2}}{3} = \frac{k+5}{-2} = \frac{-\frac{3}{2}h+5k-15}{-34}$$

after solving centre C is

$$(h, k) = \left(8, \frac{-28}{3}\right)$$

and radius of circle is

$$r = \left|\frac{3(8)-2\left(\frac{-28}{3}\right)-34}{\sqrt{9+4}}\right| = \left|\frac{24+2\frac{56}{3}-34}{\sqrt{13}}\right|$$

$$r = \left|\frac{26}{3\sqrt{13}}\right| = \frac{2\sqrt{13}}{3}$$

78. Let $f(x) = x + \frac{a}{\pi^2-4}\sin x + \frac{b}{\pi^2-4}\cos x, x \in \mathbb{R}$ be a function which satisfies $f(x) = x + \int_0^{\pi/2} \sin(x + y) f(y) dy$. Then (a+b) is equal to

(1) $-2\pi(\pi-2)$ (2) $-2\pi(\pi+2)$ (3) $-\pi(\pi-2)$ (4) $-\pi(\pi+2)$

Sol. **2**

$$f(x) = x + \int_0^{\frac{\pi}{2}} (\sin x \cos y + \cos x \sin y) f(y) dy$$

$$f(x) = x + \int_0^{\frac{\pi}{2}} (\cos y\, f(y) dy) \sin x + (\sin y\, f(y) dy) \cos x) \qquad \text{.........(1)}$$

given : $f(x) = x + \frac{a}{\pi^2 - 4}\sin x + \frac{b}{\pi^2 - 4}\cos x$ ....(2)

by comparing (1) and (2)

$$\Rightarrow \frac{a}{\pi^2-4} = \int_0^{\frac{\pi}{2}} \cos y\, f(y)dy \qquad ....(3)$$

and $$\frac{b}{\pi^2-4} = \int_0^{\frac{\pi}{2}} \sin y\, f(y)dy \qquad ....(4)$$

adding (3) and (4)

$$\frac{a+b}{\pi^2-4} = \int_0^{\frac{\pi}{2}} (\sin y + \cos y) f(y)dy \qquad ....(5)$$

$$\frac{a+b}{\pi^2-4} = \int_0^{\frac{\pi}{2}} (\sin y + \cos y) f\left(\frac{\pi}{2} - y\right)dy \qquad ....(6)$$

Additing (5) and (6)

$$\frac{2(a+b)}{\pi^2-4} = \int_0^{\frac{\pi}{2}} (\sin y + \cos y)\left(\frac{\pi}{2} + \frac{a+b}{\pi^2-4}(\sin y + \cos y)\right)dy$$

$$= \pi + \frac{a+b}{\pi^2-4}\left(\frac{\pi}{2}+1\right)$$

$\Rightarrow a + b = -2\pi(\pi + 2)$

79. Let $f: \mathbf{R} \to \mathbf{R}$ be a function such that $f(x) = \frac{x^2+2x+1}{x^2+1}$. Then

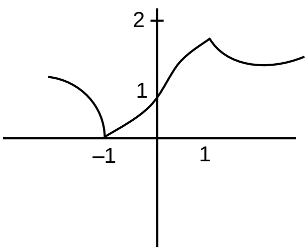
(1) $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$

(2) $f(x)$ is one-one in $(-\infty, \infty)$

(3) $f(x)$ is many-one in $(-\infty, -1)$

(4) $f(x)$ is many-one in $(1, \infty)$

Sol. **1**



$$f(x) = \frac{(x+1)^2}{x^2+1} = 1 + \frac{2x}{x^2+1}$$

$$f(x) = 1 + \frac{2}{x + \frac{1}{x}}$$

Clearly, f(x) is one – one in [1, ∞]

but not in (-∞, ∞)

80. Three rotten apples are mixed accidently with seven good apples and four apples are drawn one by one without replacement. Let the random variable X denote the number of rotten apples. If μ and σ^2 represent mean and variance of X, respectively, then $10(\mu^2+\sigma^2)$ is equal to

(1) 250 (2) 25 (3) 30 (4) 20

Sol. **4**

Total Apple = 10, Rotten apple = 3, good apple = 7

Prob. of rotten apple (p) = $\frac{3}{10}$

Prob. of good apple (q) = $\frac{7}{10}$

x $\rightarrow$ Number of rotten apples

here x = 0, 1, 2, 3

$p(x=0) = {}^4C_0\left(\frac{3}{10}\right)^0 \times \frac{7}{10}\times\frac{6}{9}\times\frac{5}{8}\times\frac{4}{7}=\frac{1}{6}$

$p(x=1) = {}^4C_1\left(\frac{3}{10}\right)\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}=\frac{1}{2}$

$p(x=2) = {}^4C_2\left(\frac{3}{10}\times\frac{2}{9}\right)\times\frac{7}{8}\times\frac{6}{7}=\frac{3}{10}$

$p(x=3) = {}^4C_3\left(\frac{3}{10}\times\frac{2}{9}\times\frac{1}{8}\right)\times\frac{7}{7}=\frac{1}{30}$

x_i	0	1	2	3
p_i	$\frac{35}{210}$	$\frac{105}{210}$	$\frac{3}{10}$	$\frac{1}{30}$

Now,

$\mu = \sum p_i x_i = \frac{1}{6}\times 0+\frac{1}{2}\times 1+2\times\frac{3}{10}+3\times\frac{1}{30}=\frac{6}{5}$

and $\sigma^2 = \sum p_i x_i^2 - \mu^2 = \frac{1}{2}+\frac{3}{10}\times 4+\frac{1}{30}\times 9-\frac{36}{25}=\frac{14}{25}$

$\therefore\ 10(\mu^2+\sigma^2) = 10\left(\frac{36}{25}+\frac{14}{25}\right)$

$= 10\times\left(\frac{50}{25}\right)=10\times 2$

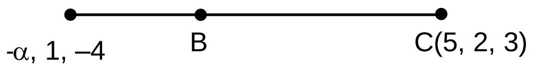
$= 20$

Section B

81. Let the co-ordinates of one vertex of $\triangle ABC$ be $A(0,2,\alpha)$ and the other two vertices lie on the line $\frac{x+\alpha}{5}=\frac{y-1}{2}=\frac{z+4}{3}$. For $\alpha\in\mathbb{Z}$, if the area of $\triangle ABC$ is 21 sq. units and the line segment BC has length $2\sqrt{21}$ units, then α^2 is equal to

Sol. **9**

A (0, 2, α)



$$\left|\frac{1}{2}.2\sqrt{21}\begin{vmatrix} i & j & k \\ \alpha & 1 & \alpha+4 \\ 5 & 2 & 3 \end{vmatrix}\frac{1}{\sqrt{25+4+9}}\right| = 21$$

$$\sqrt{(2\alpha+5)^2+(2\alpha+20)^2+(2\alpha-5)^2} = \sqrt{21}\sqrt{38}$$

$12\alpha^2 + 80\alpha + 450 = 798$

$12\alpha^2 + 80\alpha - 398 = 0$

$\alpha = 3 \Rightarrow \alpha^2 = 9$

82. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function that satisfies the relation $f(x+y) = f(x) + f(y) - 1, \forall x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to

Sol. **3**

Given $f(x + y) = f(x) + f(y) - 1 \ \forall \ x, y \in IR$ and $f'(0) = 2$

Partial differentiate w.r.t x

$\Rightarrow f'(x + y)\ f'(x)$

for $x = 0$

$f'(y) = f'(0) = 2$

on Integrating

$\Rightarrow f(y) = 2y + c$ ……...(2)

for $y = 0$

$\Rightarrow f(0) = C$ …….(3)

Put $x = y = 0$ in (1)

$\Rightarrow f(0) = f(0) + f(0) - 1$

$\Rightarrow f(0) = 1$ ……..(4)

from (3) & (4)

$c = 1$

$\Rightarrow f(y) = 2y + 1$

$\Rightarrow f(-2) = -4 + 1 = -3$

$\therefore |f(-2)| = 3$

83. Suppose f is a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{N}$ and $f(1) = \frac{1}{5}$. If $\sum_{n=1}^{m}\frac{f(n)}{n(n+1)(n+2)} = \frac{1}{12}$, then m is equal to

Sol. **10**

$f(x + y) = f(x) + f(y) \ \forall \ x, y \in N$ and $f(1) = \frac{1}{5}$

for $x = y = 1$

$f(2) = f(1) + f(1) = 2f(1)$

$f(3) = f(2+1) = f(2) + f(1) = 3f(1)$

In General

$f(n) = nf(1) = \frac{n}{5}$

$$\sum_{n=1}^{m}\frac{f(n)}{n(n+1)(n+2)}=\frac{1}{12}$$

$$\Rightarrow \sum_{n=1}^{m}\frac{n}{5n(n+1)(n+2)}=\frac{1}{12}$$

$$\Rightarrow \frac{1}{5}\sum_{n=1}^{m}\frac{1}{(n+1)(n+2)}=\frac{1}{12}$$

$$\Rightarrow \sum_{n=1}^{m}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{5}{12}$$

$$\Rightarrow \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+........+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)=\frac{5}{12}$$

$$\Rightarrow \frac{1}{2}-\frac{1}{m+2}=\frac{5}{12}$$

$$\Rightarrow \frac{1}{m+2}=\frac{1}{2}-\frac{5}{12}=\frac{1}{12}$$

$\Rightarrow$ m = 10

84. Let the coefficients of three consecutive terms in the binomial expansion of $(1+2x)^n$ be in the ratio $2:5:8$. Then the coefficient of the term, which is in the middle of these three terms, is

Sol. **1120**

Let r + 1, r + 2 and r + 3 be three consecutive terms

$$\frac{{}^nC_r 2^r}{{}^nC_{r+1}2^{r+1}}=\frac{2}{5}$$

$$\Rightarrow \frac{r+1}{n-r}=\frac{4}{5} \qquad \text{……(1)}$$

Also,

$$\frac{{}^nC_{r+1}2^{r+1}}{{}^4C_{r+2}2^{r+2}}=\frac{5}{8}$$

$$\Rightarrow \frac{r+2}{n-r-1}=\frac{5}{4} \qquad \text{…….(2)}$$

on solving (1) & (2), we get

n = 8, r = 3

Here n = 8 (even)

middle term = r + 2 = 3 + 2 = 5

coefficient of T_5 = ${}^8C_4 2^4=70(16)=1120$

85. Let $a_1, a_2, a_3, \ldots$ be a GP of increasing positive numbers. If the product of fourth and sixth terms is 9 and the sum of fifth and seventh terms is 24 , then $a_1a_9+a_2a_4a_9+a_5+a_7$ is equal to

Sol. **60**

Let first term of G.P be a with common ratio r

Given : $a_4 \cdot a_6 = 9$

$a_5 + a_7 = 24$

$a_4 = ar^3$, $a_5 = ar^4$, $a_6 = ar^5$, $a_7 = ar^6$

$a_4 \cdot a_6 = a^2r^8 = 9$

$\Rightarrow \quad ar^4 = 3$

$a_5 = 3$

$\therefore \quad a_7 = 24 - 3 = 21$

$\Rightarrow \quad \frac{a_7}{a_5} = r^2 = 7$

$\Rightarrow \quad r = \sqrt{7}, \ a = \frac{3}{49}$

$a_1 a_9 + a_2 a_4 a_9 + \quad a_5 + a_7 = a_1 a_9 + (ar)(ar^3) a_9 + 24$

$= a_1 a_9 + a_1(ar^4)a_9 + 24$

$= a_1 a_9 (1 + a_5) + 24 = (ar^4)^2 (4) + 24$

$= 36 + 24 = 60$

86. Let the equation of the plane P containing the line $x + 10 = \frac{8-y}{2} = z$ be $ax + by + 3z = 2(a + b)$ and the distance of the plane P from the point $(1,27,7)$ be c. Then $a^2 + b^2 + c^2$ is equal to

Sol. **355**

Given equation of plane is

$ax + by + 3z = 2(a + b)$ ……..(1)

It containing the line

$\frac{x-(-10)}{1} = \frac{y-8}{-2} = \frac{z-0}{1}$

$\therefore$ plane (1) must passes through (–10, 8, 0) and parallel to 1, –2, 1

Hence,

$a(-10) + 8b = 2a + 2b$

$\Rightarrow \quad 12a - 6b = 0$ ……..(2)

and $\quad a - 2b + 3 = 0$ ……..(3)

on solving (2) and (3), we get

$b = 2, a = 1$

$\therefore$ equation of the plane is

$x + 2y + 3z = 6$ ……..(4)

c is perpendicular distance from (1, 27, 7) to the plane (4)

$\Rightarrow c = \left|\frac{1+2\times27+3\times7-6}{\sqrt{1^2+2^2+3^2}}\right| = \left|\frac{70}{\sqrt{14}}\right| = \frac{10\sqrt{7}}{\sqrt{2}}$

Now, $a^2 + b^2 + c^2 = 1 + 4 + \frac{700}{2} = \frac{710}{2} = 355$

87. If the co-efficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ and the co-efficient of x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ are equal, then $(\alpha\beta)^2$ is equal to

Sol. **1**

For $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$

$T_{r+1} = {}^{11}C_r (\alpha x^3)^{11-r} \left(\frac{1}{\beta x}\right)^r$

$= {}^{11}C_r \alpha^{11-r} \beta^{-r} x^{33-4r}$

Coefficient of $x^9 = {}^{11}C_6 \alpha^{11-6} \beta^{-6}$

$= {}^{11}C_6 \alpha^5 \beta^{-6}$

For $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$

$T_{r+1} = {}^{11}C_r (\alpha x)^{11-r} \left(\frac{-1}{\beta x^3}\right)^r$

$= (-1)^r\ {}^{11}C_r \alpha^{11-r} \beta^{-r} x^{11-4r}$

coefficient of $x^{-9} = -\ {}^{11}C_5 \alpha^6 \beta^{-5}$

$\Rightarrow {}^{11}C_6 \alpha^5 \beta^{-6} = {}^{11}C_5 \alpha^6 \beta^{-5}$

$\Rightarrow \alpha\beta = -\frac{{}^{11}C_6}{{}^{11}C_5} = -1$

$\therefore (\alpha\beta)^2 = 1$

88. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero non-coplanar vectors. Let the position vectors of four points A, B, C and D be $\vec{a} - \vec{b} + \vec{c}$, $\lambda\vec{a} - 3\vec{b} + 4\vec{c}$, $-\vec{a} + 2\vec{b} - 3\vec{c}$ and $2\vec{a} - 4\vec{b} + 6\vec{c}$ respectively. If $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then λ is equal to

Sol. **2**

$\overrightarrow{AB} = (\lambda\vec{a} - 3\vec{b} + 4\vec{c}) - (\vec{a} - \vec{b} + \vec{c})$

$= (\lambda - 1)\ \vec{a} - 2\vec{b} + 3\vec{c}$

$\overrightarrow{AC} = (-\vec{a} + 2\vec{b} - 3\vec{c}) - (\vec{a} - \vec{b} + \vec{c})$

$= -2\vec{a} + 3\vec{b} - 4\vec{c}$

$\overrightarrow{AD} = (2\vec{a} - 4\vec{b} + 6\vec{c}) - (\vec{a} - \vec{b} + \vec{c})$

$= \vec{a} - 3\vec{b} + 5\vec{c}$

For coplanar vectors

$$\begin{vmatrix} \lambda - 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$$

$\Rightarrow 3\lambda - 6 = 0$

$\therefore \lambda = 2$

89. Five digit numbers are formed using the digits 1, 2, 3,5, 7 with repetitions and are written in descending order with serial numbers. For example, the number 77777 has serial number 1 . Then the serial number of 35337 is

Sol. **1436**

Number starting with 7 = $7\ \underset{5}{\underset{\uparrow}{\_}}\ \underset{5}{\underset{\uparrow}{\_}}\ \underset{5}{\underset{\uparrow}{\_}}\ \underset{5}{\underset{\uparrow}{\_}} = 625$

Number starting with 5 = $5\ \underset{5}{\underset{\uparrow}{\_}}\ \underset{5}{\underset{\uparrow}{\_}}\ \underset{5}{\underset{\uparrow}{\_}}\ \underset{5}{\underset{\uparrow}{\_}} = 625$

Number starting with 37 = $37\underset{5}{\underline{\ }}\ \underset{5}{\underline{\ }}\ \underset{5}{\underline{\ }}$ = 125

Number starting with 357 = $357\underset{5}{\underline{\ }}\ \underset{5}{\underline{\ }}$ = 25

Number starting with 355 = $355\_\ \_$ = 25

Number starting with 3537 = $3537\ \_ = 5$

Number starting with 3535 = $3535\ \_ = 5$

Number starting with 35337 = 1

Total = 1436

Therefore, the serial number of 35337 is 1436

90. If all the six digit numbers $x_1x_2x_3x_4x_5x_6$ with $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are arranged in the increasing order, then the sum of the digits in the 72^{th} number is

Sol. **32**

Number of six digit number starting with 1 is 1 ………….. = ${}^8C_5 = 56$

As remaining five digits can be selected from 8 digits that are greater than (i.e., 2, 3, 4, 5, 6, 7, 8)

Number of six digit number starting with 23 ………. = ${}^6C_4 = 15$

Total = 56 + 15 = 71

Now, 72^{nd} number = 245678

$\therefore$ sum of the digits = 2 + 4 + 5 + 6 + 7 + 8 =32


मोशन है, तो
भरोसा है।
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Session 2023-24 (English & हिन्दी Medium)
Target: JEE/NEET 2025
Nurture & प्रयास Batch
Class 10th to 11th Moving
Target: JEE/NEET 2024
Enthuse & प्रयास Batch
Class 11th to 12th Moving
Target: JEE/NEET 2024
Dropper & प्रयास Batch
Class 12th to 13th Moving
Target: PRE FOUNDATION
SIP, Evening & Tapasya Batch
Class 6th to 10th Students
Motion