JEE MAIN 2024 SESSION-2

Paper with Solution

MATHS | 06th April 2024 _ Shift-2

SECTION – A

1. If three letters can be posted to any one of the 5 different addresses, then the probability that the three letters are posted to exactly two addresses is :

(1) $\frac{18}{25}$ (2) $\frac{12}{25}$ (3) $\frac{6}{25}$ (4) $\frac{4}{25}$

Sol. **(2)**

Total out comes = 5 × 5 × 5 = 125

Favourable cases = $^5C_2 \cdot {}^2C_1 \cdot {}^3C_1$

$^5C_2$ ↓ selection of 2address from 5; $^2C_1$ ↓ selection of the address where '2' leter will go; $^3C_1$ ↓ unique address will go at letter which selection

$$P(E)=\frac{60}{125}=\frac{12}{25}$$

2. If $\int\frac{1}{a^2\sin^2 x+b^2\cos^2 x}dx=\frac{1}{12}\tan^{-1}(3\tan x)+$ constant, then the maximum value of a sin x + b cos x, is :

(1) $\sqrt{40}$ (2) $\sqrt{41}$ (3) $\sqrt{42}$ (4) $\sqrt{39}$

Sol. **(1)**

$$\int\frac{1}{a^2\sin^2 x+b^2\cos^2 x}dx=\frac{1}{12}\tan^{-1}(3\tan x)+\text{constant}$$

Divide numerator & denominator by $\cos^2 x$

$$\int\frac{\sec^2 x dx}{a^2\tan^2 x+b^2}=\frac{1}{12}\tan^{-1}(3\tan x)+\text{constant}$$

Put $a\tan x=t$

$a\sec^2 x dx=dt$

$\sec^2 x dx=\frac{1}{a}dt$

$$\frac{1}{a}\int\frac{dt}{t^2+b^2}=\frac{1}{12}\tan^{-1}(3\tan x)+\text{constant}$$

$$\frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right)=\frac{1}{12}\tan^{-1}(3\tan x)$$

Comparing $\frac{a}{b}=3$ & $ab=12$

$a=6$ & $b=2$

Maximum value of $6\sin x+2\cos x$ is $\sqrt{6^2+2^2}=\sqrt{40}$

3. If A is a square matrix of order 3 such that det(A) = 3 and $\det\left(\text{adj}\left(-4\,\text{adj}\left(-3\,\text{adj}\left(3\,\text{adj}\left((2A)^{-1}\right)\right)\right)\right)\right) = 2^m 3^n$, then m + 2n is equal to :

(1) 3 (2) 6 (3) 4 (4) 2

Sol. **(3)**

Det(A) = 3 or |A| = 3

$$= \left|-4\,\text{adj}\left(-3\,\text{adj}\left(3\,\text{adj}(2A)^{-1}\right)\right)\right|^2$$

$$= (-4)^6 \left|4\,\text{adj}\left(-3\,\text{adj}\left(3\,\text{adj}(2A)^{-1}\right)\right)\right|^2$$

$$= 2^{12}(-3)^{12}\left|\text{adj}\left(3\,\text{adj}(2A)^{-1}\right)\right|^4$$

$$= 2^{12}\cdot 3^{12}\left|3\,\text{adj}(2A)^{-1}\right|^8$$

$$= 2^{12}\cdot 3^{12}\cdot 3^{24}\left|\text{adj}(2A)^{-1}\right|^8$$

$$= 2^{12}\cdot 3^{36}\left|(2A)^{-1}\right|^{16}$$

$$\Rightarrow 2^{12}\cdot 3^{36}\frac{1}{|2A|^{16}}$$

$$\Rightarrow \frac{2^{12}.3^{36}}{2^{48}|A|^{16}}$$

$$\Rightarrow \frac{2^{12}.3^{36}}{2^{48}\cdot 3^{16}} \Rightarrow \frac{3^{20}}{2^{36}} = 2^{-36}\cdot 3^{20}$$

$m = -36$

$n = 20$

$m + 2n = -36 + 40 = 4$

4. Suppose the solution of the differential equation $\frac{dy}{dx} = \frac{(2+\alpha)x - \beta y + 2}{\beta x - 2\alpha y - (\beta\gamma - 4\alpha)}$ represents a circle passing through origin. Then the radius of this circle is :

(1) $\frac{1}{2}$ (2) $\sqrt{17}$ (3) $\frac{\sqrt{17}}{2}$ (4) 2

Sol. **(3)**

$$\frac{dy}{dx}=\frac{(2+\alpha)x-\beta y+2}{\beta x-2\alpha y-(\beta\gamma-4\alpha)}$$

$$\frac{dy}{dx}=\frac{(2+\alpha)x-\beta y+2}{\beta x-(2\alpha+\beta)y+4\alpha)}$$

$$\beta xdy-2\alpha ydy-(\beta\gamma-4\alpha)dy=(\alpha+2)xdx-\beta ydx+2dx$$

$$\beta(xdy+ydx)-(2\alpha+\beta)ydy+4\alpha dy=(2+\alpha)xdx+2dx$$

$$\beta d(xy)-(2\alpha+\beta)ydy+4\alpha dy=(\alpha+2)xdx+2dy$$

$$\beta\int d(xy)-(2\alpha+\beta)\int ydy+4\alpha\int dy=(\alpha+2)\int xdx+2\int dx$$

$$\beta xy-(2\alpha+\beta)\frac{y^2}{2}+4\alpha y=(\alpha+2)\frac{x^2}{2}+2x+c$$

Solution of D.E represent a circle

When $\beta=0$ and $\frac{\alpha+2}{2}=\frac{(2\alpha+\beta)}{2}$

$$\alpha+2=2\alpha+\beta$$

$$\boxed{\alpha=2}$$

Put $\alpha=2$ & $\beta=0$ in equation (1)

$$\frac{4}{2}x^2+\frac{4}{2}y^2+2x-8y+c=0$$

$$2x^2+2y^2+2x-8y+c=0$$

It is passes through (0,0) $\Rightarrow c=0$

$\therefore$ $x^2+y^2+x-4y=0$ Equation of circle

Radius $=\sqrt{\frac{1}{4}+4}=\frac{\sqrt{17}}{2}$

5. If all the words with or without meaning made using all the letters of the word "NAGPUR" are arranged as in a dictionary, then the word at 351th position in this arrangement is :

(1) NRAPUG (2) NRAGUP (3) NRAPGU (4) NRAGPU

Sol. **(3)**

NAGPUR

A,G,N,P,R,U

A _ _ _ _ _ 5! = 120
G _ _ _ _ _ 5! = 120
N A _ _ _ _ 4! = 24
N G _ _ _ _ 4! = 24

N P _ _ _ _ 4! = 24
N R A G P U 1 = 1
N R A G U P 1 = 1

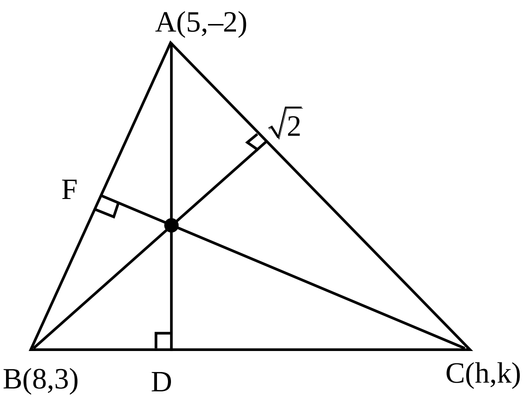
134

N R A P G U +1 = 315

6. If P(6, 1) be the orthocentre of the triangle whose vertices are A(5, –2), B(8, 3) and C (h, k), then the point C lies on the circle :

(1) $x^2 + y^2 - 65 = 0$ (2) $x^2 + y^2 - 52 = 0$ (3) $x^2 + y^2 - 61 = 0$ (4) $x^2 - y^2 - 74 = 0$

Sol. **(1)**



$m_{AD} = 3$

$M_{BC} = \frac{K-3}{h-8}$

$m_{AD} \times m_{BC} = -1$

$3 \times \frac{K-3}{h-8} = -1$

$h + 3K - 17 = 0$ ...(1)

$h + 3K - 17 = 0$

$m_{AC} = \frac{K+2}{h-5}$

$m_{BE} \times m_{AC} = -1$

$h + K - 3 = 0$ ....(2)

Solve (1) & (2)

$h = -4, \; K = 7$

Hence c(–4, 7)

7. Let $f(x) = \dfrac{1}{7 - \sin 5x}$ be a function defined on **R**. Then the range of the function f(x) is equal to :

(1) $\left[\frac{1}{7}, \frac{1}{5}\right]$ (2) $\left[\frac{1}{8}, \frac{1}{5}\right]$ (3) $\left[\frac{1}{8}, \frac{1}{6}\right]$ (4) $\left[\frac{1}{7}, \frac{1}{6}\right]$

Sol. **(3)**

$f(x) = \dfrac{1}{7 - \sin 5x}$

$\sin 5x \in [-1, 1]$

$-\sin 5x \in [-1, 1]$

$7 - \sin 5x \in [6, 8]$

$\dfrac{1}{7 - \sin 5x} \in \left[\frac{1}{8}, \frac{1}{6}\right]$

$y \in \left[\frac{1}{8}, \frac{1}{6}\right]$

8. Let $0 \le r \le n$. If ${}^{n+1}C_{r+1} : {}^{n}C_r : {}^{n-1}C_{r-1} = 55 : 35 : 21$, then 2n + 5r is equal to:

(1) 55 (2) 60 (3) 62 (4) 50

Sol. **(4)**

$${}^{n+1}C_{r+1} : {}^{n}C_r : {}^{n-1}C_{r-1} = 55 : 35 : 21$$

$$\frac{{}^{n+1}C_{r+1}}{55} = \frac{{}^{n}C_r}{35} = \frac{{}^{n-1}C_{r-1}}{21}$$

$$\frac{\frac{(n+1)n}{(r+1)r}\,{}^{n-1}C_{r-1}}{55} = \frac{\frac{n}{r}\,{}^{n-1}C_{r-1}}{35} = \frac{{}^{n-1}C_{r-1}}{21}$$

$$\underbrace{\frac{n(n+1)}{55r(r+1)} = \frac{n}{35r}} = \frac{1}{21}$$

$$\frac{(n+1)}{(r+1)}=\frac{55}{35}\Rightarrow\frac{n+1}{r+1}=\frac{11}{7}$$

$$7n+7=11r+11$$

$$7n-11r=4 \qquad ....(1)$$

$$\frac{n}{35r}=\frac{1}{21}\Rightarrow\frac{n}{5r}=\frac{1}{3}$$

$$3n-5r=0 \qquad ....(2)$$

Solve (1) and (2) $n=\frac{5r}{3}$

$$7\times\frac{5r}{3}-11r=4$$

$$r=6 \;\;\&\;\; n=\frac{5\times 6}{3} \qquad \Rightarrow n=10$$

$$2n+5r=2\times 10+5\times 6 \;=50$$

9. If z_1, z_2 are two distinct complex number such that $\left|\frac{z_1-2z_2}{\frac{1}{2}-z_1\bar{z}_2}\right|=2$, then

(1) z_1 lies on a circle of radius $\frac{1}{2}$ and z_2 lies on a circle of radius 1 .

(2) either z_1 lies on a circle of radius 1 or z_2 lies on a circle of radius $\frac{1}{2}$.

(3) either z_1 lies on a circle of radius $\frac{1}{2}$ or z_2 lies on a circle of radius 1 .

(4) both z_1 and z_2 lie on the same circle.

Sol. **(2)**

$$\left|\frac{z_1-2z_2}{\frac{1}{2}-z_1\bar{z}_2}\right|=2$$

$$\Rightarrow \qquad |z_1-2z_2|=|1-2z_1\bar{z}_2|$$

$$\Rightarrow \qquad |z_1-2z_2|^2=|1-2z_1\bar{z}_2|^2$$

$$\Rightarrow \qquad (z_1-2z_2)(\bar{z}_1-2\bar{z}_2)=(1-2z_1\bar{z}_2)(1-2\bar{z}_1z_2)$$

$\Rightarrow \quad |z_2|^2 - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + 4|z_2|^2 = 1 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + 4|z_1|^2|z_2|^2$

$\Rightarrow \quad |z_1|^2 - 1 + 4|z_2|^2 - 4|z_1|^2|z_2|^2 = 0$

$\Rightarrow \quad \left(|z|^2 - 1\right) - 4|z_2|^2\left(|z_1|^2 - 1\right) = 0$

$\Rightarrow \quad \left(|z_1|^2 - 1\right)\left(1 - 4|z_2|^2\right) = 0$

either $|z_1|^2 - 1 = 0$ or $1 - 4|z_2|^2 = 0$

$|z_1| = 1$ or $|z_2| = \frac{1}{2}$

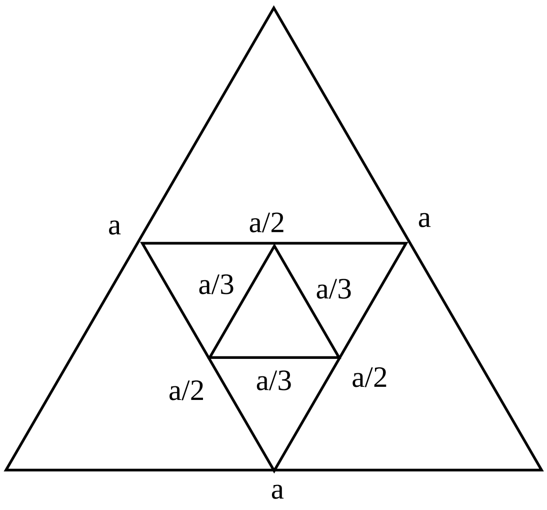
$\therefore$ either z_1 lies on circle of radius 1

or z_2 lies on circle of radius $\frac{1}{2}$

10. Let ABC be an equilateral triangle. A new triangle is formed by joining the middle points of all sides of the triangle ABC and the same process is repeated infinitely many times. If P is the sum of perimeters and Q is be the sum of areas of all the triangles formed in this process, then :

(1) $P^2 = 36\sqrt{3}Q$ (2) $P^2 = 72\sqrt{3}Q$ (3) $P = 36\sqrt{3}Q^2$ (4) $P^2 = 6\sqrt{3}Q$

Sol. **(1)**



Area of first equilateral $\Delta = \frac{\sqrt{3}}{4}a^2$

Area of Second equilateral $\Delta = \frac{\sqrt{3}}{4}\left(\frac{a}{2}\right)^2 = \frac{\sqrt{3}a^2}{16}$

Area of Second equilateral $\Delta = \frac{\sqrt{3}}{4}\left(\frac{a}{4}\right)^2 = \frac{\sqrt{3}a^2}{64}$

Sum of area $= \frac{\sqrt{3}}{4}a^2 + \frac{\sqrt{3}}{16}a^2 + \frac{\sqrt{3}}{64}a^2 + ...$

$$= \frac{\sqrt{3}a^2}{4}\left(1 + \frac{1}{4} + \frac{1}{16} + ...\right)$$

$$= \frac{\sqrt{3}}{4}a^2\left(\frac{1}{1-\frac{1}{4}}\right)$$

$$Q = \frac{\sqrt{3}}{4}a^2\left(\frac{4}{3}\right)$$

$$Q = \frac{a^2}{\sqrt{3}} \quad ....(1)$$

Perimeter of first $D = 3a$

Perimeter of second $D = \frac{a}{2} + \frac{a}{2} + \frac{a}{2} = \frac{3a}{2}$

Perimeter of third $D = \frac{a}{4} + \frac{a}{4} + \frac{a}{4} = \frac{3a}{4}$

Sum of perimeter P = $P = 3a + \frac{3a}{2} + \frac{3a}{4} + ...$

$$P = 3a(1 + \frac{1}{2} + \frac{1}{4} + ...)$$

$$P = \frac{3a}{1-\frac{1}{2}} = 6a$$

$$P^2 = 36a^2 \quad .....(2)$$

From (1) & (2)

$$\frac{Q}{P^2} = \frac{\alpha^2}{\sqrt{3}.36\alpha^2}$$

$$P^2 = 36\sqrt{3}Q$$

11. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}, \vec{b} = ((\vec{a} \times (\hat{i} + \hat{j})) \times \hat{i}) \times \hat{i}$. Then the square of the projection of $\vec{a}$ on $\vec{b}$ is :

(1) $\frac{1}{3}$ (2) 2 (3) $\frac{1}{5}$ (4) $\frac{2}{3}$

Sol. (2)

$\bar{a} = 2\hat{i} + \hat{j} - \hat{k}$ $\bar{b} = \left(\left(\bar{a} \times (\hat{i} + \hat{j}) \times \hat{i}\right)\right) \times \hat{i}$

$$\bar{a} \times \left(\hat{i} + \hat{j}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k}$$

$$\left(\bar{a} \times \left(\hat{i} + \hat{j}\right)\right) \times \hat{i} = \hat{k} + \hat{j}$$

$$\left(\bar{a} \times \left(\hat{i} + \hat{j}\right)\right) \times \hat{i} = (\hat{k} + \hat{j}) \times \hat{i}$$

$$= \hat{j} - \hat{k}$$

Projection of $\bar{a}$ on $\bar{b}$ $= \frac{\bar{a}.\bar{b}}{|\bar{b}|} = \frac{\left(2\hat{i} + \hat{j} - \hat{k}\right)\left(\hat{i} - \hat{k}\right)}{\left|\hat{i} - \hat{k}\right|}$

$$= \frac{1+1}{\sqrt{2}} = \sqrt{2}$$

Square of projection of $\bar{a}$ or $\bar{b}$ is 2

12. If the locus of the point, whose distances from the point (2,1) and (1,3) are in the ratio 5: 4, is ax^2,+ b y^2 + cxy + d x + ey +170 = 0, then the value of a^2 + 2 b + 3 c + 4 d + e is equal to:

(1) 437 (2) –27 (3) 5 (4) 37

Sol. (4)

Let point P(x,y) and A(2,1) & B(1,3)

According to question

$$\frac{PA}{PB} = \frac{5}{4}$$

$$\frac{(x-2)^2 + (y-1)^2}{(x-1)^2 + (y-3)^2} = \frac{25}{16}$$

$$16x^2 + 16y^2 - 64x + 64 - 32y + 16 = 25x^2 + 25y^2 - 50x - 150y + 250$$

$$9x^2 + 9y^2 + 14x - 118y + 170 = 0$$

Now compare with

$$ax^2+by^2+cxy+dx+ey+170=0$$

$$a=9, b=9, c=0, d=14, e=-118$$

$$a^2+2b+3c+4d+e=81+18+0+56-118=37$$

13. Let A = {1,2,3,4,5}. Let R be a relation on A defined by xRy if and only if 4x ≤ 5y. Let m be the number of elements in R and n be the minimum number of elements from A × A that are required to be added to R to make it a symmetric relation. Then m + n is equal to :

(1) 24 (2) 23 (3) 26 (4) 25

Sol. **(4)**

$A=\{1,2,3,4,5\}$

$R=\{(1,1)(1,2)(1,3)(1,4)(1,5)(2,2)(2,3)(2,4)$
$(2,5)(3,3)(3,4)(3,5)(4,4)(4,5)(5,4)(5,5)\}$

m = 16

Now to make R a symmetric relation add

$\{(2,1)(3,1)(4,1)(5,1)(3,2)(4,2)(5,2)(4,3)(5,3)\}$

$n=2$

So $m+n=16+9=25$

14. Let $\vec{a}=6\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. If $\vec{c}$ is a is vector such that $|\vec{c}|\geq 6, \vec{a}\cdot\vec{c}=6|\vec{c}|, |\vec{c}-\vec{a}|=2\sqrt{2}$ and the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is 60°, then $|(\vec{a}\times\vec{b})\times\vec{c}|$ is equal to :

(1) $\frac{9}{2}(6+\sqrt{6})$ (2) $\frac{9}{2}(6-\sqrt{6})$ (3) $\frac{3}{2}\sqrt{3}$ (4) $\frac{3}{2}\sqrt{6}$

Sol. **(1)**

$|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}||\vec{c}|\sin 60°$

$=|\vec{a}\times\vec{b}||\vec{c}|\frac{\sqrt{3}}{2}$

$=|\vec{a}\times\vec{b}|=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 6 & 1 & -1\\ 1 & 1 & 0\end{vmatrix}=|\hat{i}-\hat{j}+5\hat{k}|$

$=|\vec{a}\times\vec{b}|=\sqrt{27}$

$|\vec{c}-\vec{a}|=2\sqrt{2}$ $\Rightarrow |\vec{c}|^2+|\vec{a}|^2--2\vec{c}\cdot\vec{a}=8$

$c^2+38-12|\vec{c}|-=8$

$|\vec{c}|^2-12|\vec{c}|+30=0$

$|\vec{c}|=\frac{12\pm2\sqrt{6}}{2}=6\pm\sqrt{6}$

$|\vec{c}|=6+\sqrt{6}$

Now $\left|\left(\vec{a}\times\vec{b}\right)\times\vec{c}\right|=\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\frac{\sqrt{3}}{2}$

$=\sqrt{27}\times\left(6+\sqrt{6}\right)\times\frac{\sqrt{3}}{2}$

$3\sqrt{3}(6+\sqrt{6})\frac{\sqrt{3}}{2}$

$=\frac{9}{2}\left(6\sqrt{6}\right)$

15. A software company sets up m number of computer systems to finish an assignment in 17 days. If 4 computer systems crashed on the start of the second day, 4 more computer systems crashed on the start of the third day and so on, then it took 8 more days to finish the assignment. The value of m is equal to :

(1) 150 (2) 160 (3) 125 (4) 180

Sol. **(1)**

$17m=\underset{1^{st}\text{ day}}{\underset{\downarrow}{m}}+\underset{2^{nd}\text{ day}}{\underset{\downarrow}{(m-4)}}+(m-8)+(m-12)+......+(m-4\times24)$

$17m=25-4(1+2+3+....+24)$

$17m=25-\frac{4\times24\times25}{2}$

$8m=25\times48$

$m=25\times6$

$m=150$

16. If the function $f(x)=\left(\frac{1}{x}\right)^{2x}; x>0$ attains the maximum value at $x=\frac{1}{e}$ then :

(1) $e^{2\pi}<(2\pi)^{e}$ (2) $e^{\pi}>\pi^{e}$ (3) $(2e)^{\pi}>\pi^{(2e)}$ (4) $e^{\pi}<\pi^{e}$

Sol. **(2)**

$f(x)=\left(\frac{1}{x}\right)^{2x}$

Take log both the side

$\ln f(x)=2x\ln\left(\frac{1}{x}\right)$

$\ln f(x)=-2x\ln x$

$\frac{1}{f(x)}f'(x)=-2\left[x\times\frac{1}{x}+\ln x.1\right]$

$f'(x)=-2f(x)(1+\ln x)]$

$f'(x)=-2\left(\frac{1}{x}\right)^{2x}(1+\ln x)$

$\because x>0$

$f'(x)<0$

$f(x)$ is decreasing

$f'(x)=0$

$-2\left(\frac{1}{x}\right)^{2x}(1+\ln x)=0$

$1+\ln x=0$

$\ln x=-1 \Rightarrow x=\frac{1}{e}$

+ | –

$\frac{1}{e}$

$\because f(x)$ is decreasing

When $x>\frac{1}{e}$

$e<\pi$

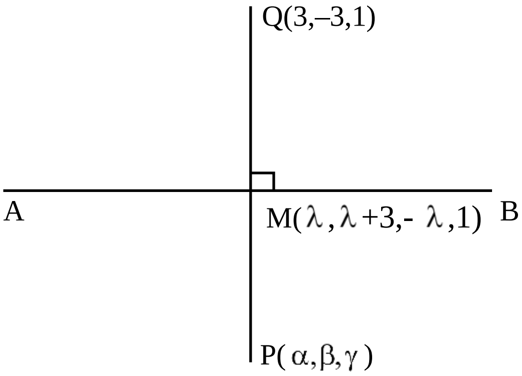
$\left(\frac{1}{e}\right)^{2e}<\left(\frac{1}{\pi}\right)^{2\pi}$

$e^{\pi}>\pi^{e}$

17. Let $P(\alpha,\beta,\gamma)$ be the image of the point $Q(3,-3,1)$ in the line $\frac{x-0}{1}=\frac{y-3}{1}=\frac{z-1}{-1}$ and R be the point (2,5,–1). If the area of the triangle PQR is λ and $\lambda^2 = 14$ K, then K is equal to :

(1) 18 (2) 72 (3) 81 (4) 36

Sol. **(3)**



$$\frac{x-0}{1}=\frac{y-3}{1}=\frac{z-1}{-1}=\lambda$$

Point M on line $M(\lambda,\lambda+3,-\lambda+1)$

Direction ratio of $QM=\lambda-3,\lambda+6,-\lambda$

$AB \perp PQ$

$\therefore\ (\lambda-3)1+(\lambda+6)+\lambda=0$

$3\lambda+3=0$

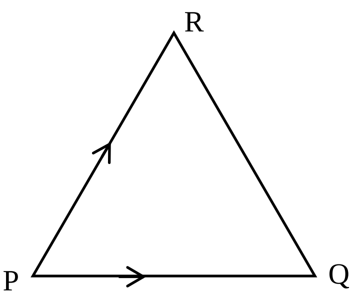
$\lambda=-1$

M is mid point of P & Q

$\frac{\alpha+3}{2}=-1$ $\frac{\beta-3}{2}=2$ $\frac{\gamma+1}{2}=2$

$\alpha=-5$ $\beta=7$ $\gamma=3$

$P(-5,7,3)$ $Q(3,-3,1)$ & $R(2,5,-1)$



Area of $\Delta PQR = \frac{1}{2}\left|\overline{PQ}\times\overline{PR}\right|$

$= \frac{1}{2}\left|(8\hat{i}-10\hat{j}-2\hat{k})\right|\times(7\hat{i}-2\hat{j}-4\hat{k})$

$= \left|\left(4\hat{i}-5\hat{j}-\hat{k}\right)\times\left(7\hat{i}-2\hat{j}-4\hat{k}\right)\right|$

$= 9\left|\left(2\hat{i}+\hat{j}+3\hat{k}\right)\right|$

$= 9\sqrt{14}$

Given that Area of ΔPQR is λ

$\lambda = 9\sqrt{14}$ $\Rightarrow \lambda^2 = 81\times 14$

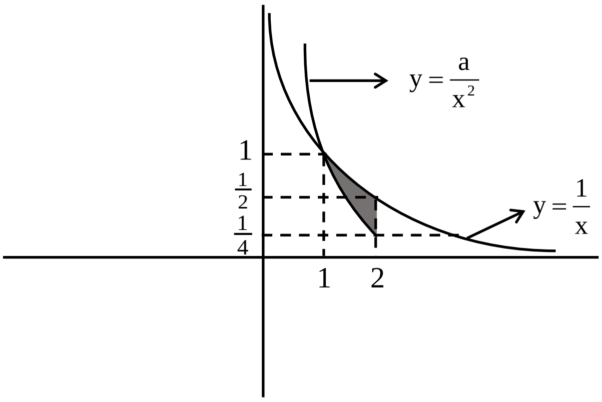
Now $\lambda^2 = 14K$ $\Rightarrow 14K = 81\times 14$

$K = 81$

18. If the area of the region $\left\{(x,y): \frac{a}{x^2} \le y \le \frac{1}{x}, 1 \le x \le 2, 0 < a < 1\right\}$ is $(\log_e 2) - \frac{1}{7}$ then the value of 7a – 3 is equal to:

(1) –1 (2) 0 (3) 2 (4) 1

Sol. **(1)**



Area of the region = $\int_1^2 \left(\frac{1}{x} - \frac{a}{x^2}\right)dx = \ln 2 - \frac{1}{7}$

$\left[\ln x + \frac{a}{x}\right]_1^2 = \ln 2 - \frac{1}{7}$

$\ln 2 + \frac{a}{2} - 0 - a = \ln 2 - \frac{1}{7}$

$$-\frac{a}{2}=-\frac{1}{7}$$

$$a=\frac{2}{7}$$

$$\therefore 7a-3=2-3=-1$$

19. $\lim\limits_{n\to\infty}\dfrac{\left(1^2-1\right)(n-1)+\left(2^2-2\right)(n-2)+\cdots\cdots+\left((n-1)^2-(n-1)\right)\cdot 1}{\left(1^3+2^3+\cdots\cdots+n^3\right)-\left(1^2+2^2+\cdots\cdots+n^2\right)}$ is equal to

(1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) $\frac{3}{4}$ (4) $\frac{1}{3}$

Sol. **(4)**

$$\lim_{n\to\infty}\frac{\sum_{r=1}^{n-1}\left(r^2-r\right)(n-r)}{(1^3+2^3+........+n^3)-(1^2+2^2+.......+n^2)}$$

$$\lim_{n\to\infty}\frac{(n+1)\sum_{r=1}^{n-1}r^2-\sum_{r=1}^{n-1}r^3-n\sum_{r=1}^{n-1}r}{\left(\frac{n(n+1)}{2}\right)^2-\frac{n(n+1)(2n+1)}{6}}$$

$$\lim_{n\to\infty}\frac{\frac{(n+1)n(n-1)(2n-1)}{6}-\frac{n^2(n-1)^2}{4}-\frac{n.n(n-1)}{2}}{\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{6}}$$

Take highest power of n common in numerator & denominator

$$\frac{\frac{2}{6}-\frac{1}{4}}{\frac{1}{4}}=\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{4}}=\frac{1}{3}$$

20. Suppose for a differentiable function h, h(0) = 0, h(1) = 1 and h'(0) = h'(1) = 2. If $g(x) = h(e^x)\,e^{h(x)}$, then g'(0) is equal to:

(1) 4 (2) 3 (3) 5 (4) 8

Sol. **(1)**

$$g(x)=h(e^x)e^{h(x)}$$

Differentiate both the side wrt ‘x’

$g'(x) = h(e^x)e^{h(x)}h'(x) + e^{h(x)}h'(e^x)e^x$

$g'(0) = h(1)e^{h(0)}h'(0) + e^{h(0)}h'(1)e^0$

$g'(0) = 1\times1\times2 + 1\times2$

$g'(0) = 4$

21. If the shortest distance between the lines $\frac{x-\lambda}{3} = \frac{y-2}{-1} = \frac{z-1}{1}$ and $\frac{x+2}{-3} = \frac{y+5}{2} = \frac{z-4}{4}$ is $\frac{44}{\sqrt{30}}$, then the largest possible value of $|\lambda|$ is equal to ____ .

Sol. **43**

$\bar{r} = (\lambda\hat{i} + 2\hat{j} + \hat{k}) + s(3\hat{i} - \hat{j} + \hat{k})$

$\bar{r} = (-2\hat{i} - 5\hat{j} + 4\hat{k}) + t(-3\hat{i} + 2\hat{j} + 4\hat{k})$

Compare with

$\bar{r} = \bar{a}_1 + s\bar{b}_1$ & $\bar{r} = \bar{a}_2 + t\bar{b}_2$

$\bar{a}_1 = \lambda\hat{i} + 2\hat{j} + \hat{k}$ $\quad\quad$ $\bar{a}_2 = -2\hat{i} - 5\hat{j} + 4\hat{k}$

$\bar{b}_1 = 3\hat{i} - \hat{j} + \hat{k}$ $\quad\quad$ $\bar{b}_2 = -3\hat{i} + 2\hat{j} + 4\hat{k}$

$$S.D = \left|\frac{(\bar{a}_1 - \bar{a}_2)\cdot(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$$

$$S.D = \left|\frac{\left((\lambda+2)\hat{i} + 7\hat{j} - 3\hat{k}\right)\cdot\left(-6\hat{i} - 15\hat{j} + 3\hat{k}\right)}{\left|-6\hat{i} - 15\hat{j} + 3\hat{k}\right|}\right|$$

$$SD = \frac{|-6\lambda - 12 - 105 - 9|}{\sqrt{36 + 225 + 9}}$$

$$\frac{|6\lambda + 126|}{3\sqrt{30}} = \frac{44}{\sqrt{30}}$$

$|6\lambda + 126| = 132$

$$6\lambda + 126 = \pm 132 \begin{cases} 6\lambda + 126 = 132 \\ \lambda = 1 \\ 6\lambda + 126 = -132 \\ \lambda = -43 \end{cases}$$

$|\lambda| = |-43| = 43$

22. Let [t] denote the greatest integer less than or equal to t. Let $f:[0,\infty)\to\mathbf{R}$ be a function defined by $f(x)=\left[\frac{x}{2}+3\right]-[\sqrt{x}]$. Let S be the set of all points in the interval [0,8] at which f is not continuous. Then $\sum_{a\in S} a$ is equal to _____ .

Sol. **17**

$$f(x)=\left[\frac{x}{2}+3\right]-[\sqrt{x}]$$

$$=\left[\frac{x}{2}\right]-\left[\sqrt{x}\right]+3 \qquad \because x\in[0,8],\ \frac{x}{2}\in[0,4],\ \sqrt{x}\in\left[0,2\sqrt{2}\right]$$

Point of discontinuity

When $\frac{x}{2}$ is integer & $\sqrt{x}$ is int.

$\frac{x}{2}=0,1,2,3,4$ $\qquad$ $\sqrt{x}=0,1,2,3$

$x=\underset{\text{Right cont}}{0},\underbrace{2,4,6,8}_{\text{discont}}$ $\qquad$ $x=\underset{\text{cont}}{0},\underbrace{1,4\ 9}_{\text{discont}}$

$\therefore$ f(x) is discontinuous at $x=1,2,6,8$

Check continuity at $x=4$

$$f(4^+)=\left[2^+\right]-\left[\sqrt{4^+}\right]+3=3$$

$$f(4^-)=\left[2^-\right]-\left[\sqrt{4^-}\right]+3=1-1+3=3$$

$$f(4)=[2]-\left[\sqrt{4}\right]+3=3$$

$\therefore f(x)$ is continuous at x = 4

Point of discontinuity = 1,2,6,8

$$\sum_{a\in s} a=1+2+6+8=17$$

23. If S(x) = (1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + …. + 60(1 + x)^{60}, x ≠ 0, and (60)^2 S(60) = a(b)^b + b, where a, b ∈ N}, then (a + b) equal to ____ .

Sol. **3660**

$S(x) = (1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \ldots + 60(1 + x)^{60}$

$$(1+x)S(x) = \quad (1+x)^2 \; +2(1+x)^3 + \;.....\; +59(1+x)^{60} + 60(1+x)^{61}$$

$$- \quad - \quad - \quad - \quad -$$

$$S(x)\{1-(1+x)\} = (1+x)+(1+x)^2+(1+x)^3+----+(1+x)^{60}-60(1+x)^{61}$$

$$S(x)(-x) = \frac{(1+x)\{(1+x)^{60}-1\}}{(1+x)-1} - 60(1+x)^{61}$$

$$-x^2S(x) = (1+x)^{61} - (1+x) - x\,60(1+x)^{61}$$

$$x^2S(x) = 60x(1+x)^{61} + (1+x) - (1+x)^{61}$$

Put x = 60

$$(60)^2S(60) = (60)^2(61)^{61} + 61 - (61)^{61}$$

$$= (61)^{61}((60)^2 - 1) + 61$$

$$(60)^2\,S(60) = (61)^{61}\,(61\times 59) + 61$$

Now compare

$$a(b)^b + b = (61\times 59)(61)^{61} + 61$$

Compare $b = 61$ & $a = 61 \times 59$

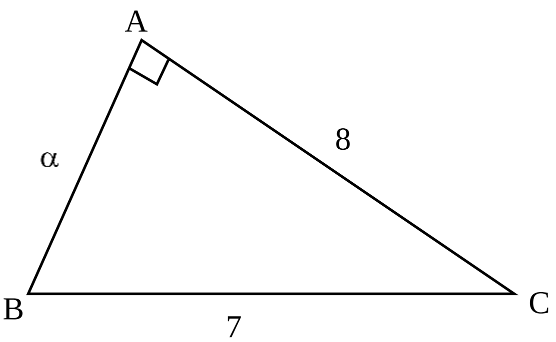
$$a + b = 61\times 59 + 61$$

$$= 61(59+1) \; = 61\times 60$$

$$a + b = 3660$$

24. In a triangle ABC, BC = 7, AC = 8, AB = $\alpha \in$ N and $\cos A = \frac{2}{3}$. If $49\cos(3C) + 42 = \frac{m}{n}$, where gcd (m, n) = 1, then m + n is equal to ______ .

Sol. **39**



$\cos A = \frac{2}{3} \quad \Rightarrow \frac{\alpha^2 + 64 - 49}{16\alpha} = \frac{2}{3}$

$3\alpha^2 - 32\alpha + 45 = 0$

$3\alpha^2 - 27\alpha - 5\alpha + 45 = 0$

$3\alpha(\alpha - 9) - 5(\alpha - 9) = 0$

$\alpha = 9, \frac{5}{3}$

But $\alpha \in N \ \therefore \alpha = 9$

Now $\cos C = \frac{49 + 64 - 81}{2 \times 7 \times 8}$

$\cos C = \frac{131 - 81}{14 \times 8} = \frac{2}{7}$

$\because 49\cos 3C + 42 = \frac{m}{n}$

$49(4\cos^3 C - 3\cos C) + 42 = \frac{m}{n}$

$49\left(\frac{32}{49 \times 7} - \frac{3 \times 2}{7}\right) + 42 = \frac{m}{n}$

$\frac{32}{7} - 7 \times 6 + 42 = \frac{m}{n}$

$\frac{m}{n} = \frac{32}{7}$

$m + n = 39$

25. If the system of equations

$2x + 7y + \lambda z = 3$

$3x + 2y + 5z = 4$

$x + \mu y + 32z = -1$

has infinitely many solutions, then $(\lambda - \mu)$ is equal to ____ .

Sol. **38**

$2x + 7y + \lambda z = 3$

$3x + 2y + 5z = 4$

$x+\mu y+32z=-1$

has infinitely many solution

$D=D_x=D_y=D_z=0$

$$D_z=\begin{vmatrix}2 & 7 & 3\\ 3 & 2 & 4\\ 1 & \mu & -1\end{vmatrix}=0$$

$-4+9\mu+28-6-8\mu+21=0$

$\mu=-39$

$$D=\begin{vmatrix}2 & 7 & \lambda\\ 3 & 2 & 5\\ 1 & -39 & 32\end{vmatrix}=0$$

$\Rightarrow \lambda=-1$

$\therefore \lambda-\mu=-1+39=38$

26. Let [t] denote the largest integer less than or equal to t.

If $\int_0^3\left(\left[x^2\right]+\left[\frac{x^2}{2}\right]\right)dx=a+b\sqrt{2}-\sqrt{3}-\sqrt{5}+c\sqrt{6}-\sqrt{7}$, where a,b,c $\in$ Z, then a + b + c is equal to

Sol. **23**

$$\int_0^3\left(\left[x^2\right]+\int_0^3\left[\frac{x^2}{2}\right]\right)dx=a+b\sqrt{2}-\sqrt{3}-\sqrt{5}+c\sqrt{6}-\sqrt{7}$$

$$\int_0^1 0dx+\int_1^3 1dx+\int_{\sqrt{2}}^{\sqrt{3}} 2dx+\int_{\sqrt{3}}^{2} 3dx+\int_2^{\sqrt{5}} 4dx+\int_{\sqrt{5}}^{\sqrt{6}} 5dx+\int_{\sqrt{6}}^{\sqrt{7}} 6dx+\int_{\sqrt{7}}^{\sqrt{8}} 7dx$$

$$+\int_{\sqrt{8}}^{3} 8dx+\int_0^{\sqrt{2}} 0dx+\int_{\sqrt{2}}^{2} 1dx+\int_2^{\sqrt{6}} 2dx+\int_{\sqrt{6}}^{\sqrt{8}} 3dx+\int_{\sqrt{8}}^{3} 4dx \;=a+b\sqrt{2}-\sqrt{3}-\sqrt{5}+c\sqrt{6}-57$$

$$31-6\sqrt{2}-\sqrt{3}-\sqrt{5}=a+b\sqrt{2}-\sqrt{3}-\sqrt{5}+c\sqrt{6}-\sqrt{7}-2\sqrt{6}-\sqrt{7}$$

Compare

$a=31,\quad b=-6,\quad c=-2$

$a+b+c=31-6-2=23$

27. If the solution y(x) of the given differential equation $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ passes through the point $\left(\frac{\pi}{2}, 0\right)$, then the value of $e^{y\left(\frac{\pi}{6}\right)}$ is equal to____.

Sol. **3**

$$(e^y+1)\cos x dx + e^y \sin x dy = 0$$

$$\cot x dx + \frac{e^y}{e^y+1}dy = 0$$

$$\int \cot x dx + \int \frac{e^y}{e^y+1}dy = 0$$

$$\ln|\sin x| + \ln|e^y+1| = \ln c$$

$$|\sin x(e^y+1)| = c$$

Put $x = \frac{\pi}{2}$ & $y = 0$

c =2

$$\sin x(e^y+1) = 2$$

$$e^y + 1 = 2\operatorname{cosec} x$$

$$e^y = 2\operatorname{cosec} x - 1$$

$$e^{y\left(\frac{\pi}{6}\right)} = 2\operatorname{cosec}\frac{\pi}{6} - 1$$

$$= 2\times 2 - 1 = 3$$

28. Let α, β be roots of $x^2 + \sqrt{2}x - 8 = 0$. If $U_n = \alpha^n + \beta^n$, then $\frac{U_{10} + \sqrt{2}U_9}{2U_8}$ is equal to____.

Sol. **4**

From Newtons identity

$$U_{10} + \sqrt{2}U_9 - 8U_8 = 0$$

$$U_{10} + \sqrt{2}U_9 = 8U_8$$

$$\frac{U_{10} + \sqrt{2}U_9}{2U_8} = 4$$

29. The length of the latus rectum and directrices of a hyperbola with eccentricity e are 9 and $x = \pm\frac{4}{\sqrt{3}}$, respectively. Let the line $y - \sqrt{3}x + \sqrt{3} = 0$ touch this hyperbola at (x_0, y_0). If m is the product of the focal distances of the point (x_0, y_0), then $4e^2 + m$ is equal to ______ .

Sol. **Bonus**

$\frac{2b^2}{a} = 9$ and $\pm\frac{a}{e} = \pm\frac{4}{\sqrt{3}}$ ... (1)

Equation of tangent $y = \sqrt{3}x - \sqrt{3}$

$m = \sqrt{3}$

$c = -\sqrt{3}$

Condition of tangency $c = \pm\sqrt{a^2m^2 - b^2}$

$-\sqrt{3} = \sqrt{3a^2 - b^2}$

$3 = 3a^2 - b^2$

$6a^2 - 9a - 6 = 0$

$2a^2 - 3a - 2 = 0$

$2a^2 - 4a + a - 2 = 0$

$(a-2)(2a+1) = 0$

$a = 2$ or $-\frac{1}{2}$

When $a = 2, b^2 = 9$ put $a = 2$ in ...(1)

$\frac{2}{e} = \frac{4}{\sqrt{3}} \Rightarrow e = \frac{\sqrt{3}}{2}$

Which is impossible

So question is bonus.

30. From a lot of 12 items containing 3 defectives, a sample of 5 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Let items in the sample be drawn one by one without replacement. If variance of X is $\frac{m}{n}$, where gcd (m, n) =1, then n –m is equal to _____ .

Sol. 71

$$P(x=0)=\frac{{}^{3}C_0\,{}^{9}C_5}{{}^{12}C_5}=\frac{9.8.7.6.5}{12.11.10.9.8}=\frac{7}{44}$$

$$P(x=1)=\frac{{}^{3}C_1\,{}^{9}C_4}{{}^{12}C_5}=\frac{3.7}{44}=\frac{21}{44}$$

$$P(x=2)=\frac{{}^{3}C_2\,{}^{9}C_3}{{}^{12}C_5}=3\times\frac{7}{66}=\frac{21}{66}$$

$$P(x=3)=\frac{{}^{3}C_3\,{}^{9}C_2}{{}^{12}C_5}=\frac{1}{22}$$

Mean $=\sum p_i x_i$

$$=P(x=0)\times 0+P(x=1).L+P(x=2).2+P(x=3).3$$

$$=0.\frac{7}{44}+\frac{21}{44}+20\times\frac{21}{66}+\frac{3}{22}$$

$$=\frac{21}{44}+\frac{7}{11}+\frac{3}{22}$$

$$=\frac{21+28+6}{44}$$

$$=\frac{55}{44}=\frac{5}{4}=1.25$$

Variance $\sigma^2=\sum p_i x_i{}^2-\mu^2$

$$=\frac{21}{44}\times 1+\frac{7}{22}\times 4+\frac{1}{22}\times 9-\frac{25}{16}$$

$$=\frac{21+56+18}{44}-\frac{25}{16}$$

$$\frac{95}{44}-\frac{25}{16}=\frac{105}{176}$$

Variance $=\dfrac{105}{176}$

$$n-m=176-105=71$$

REVISED